MATHEMATICS

AXIAL AUTOMORPHISMS OF DESIGNS. *)

BY

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1. INTRODUCTION

For a set K of integers, a $t-(v, K, 1)$ design $\mathscr{D}$ consists of a set X of points, together with certain proper subsets called blocks (or lines, if $t=2$), satisfying: each t-set of points is in a unique block; and each block has cardinality belonging to K and at least t. We wish to study such geometries having many axial automorphisms, i.e., automorphisms whose sets of fixed points are blocks. The results are as follows.

THEOREM 1. Let $\mathscr{D}$ be a $3-(v, K, 1)$ design, with K the set of even integers. Suppose $G \leqslant \mathrm{Aut}\, \mathscr{D}$ satisfies: for any three points x, y, z, their stabilizer G_{xyz} fixes all points of the block through them and has even order; and the set of fixed points of each involution in G is contained in a block. Then one of the following holds.

(i) $\mathscr{D}$ is a Miquelian inversive plane of order k, $v=k^2+1$, and $G \geqslant PGL(2, k^2)\langle t\rangle$ with t an inversion.
(ii) $\mathscr{D}$ is $AG(3, 2)$, and G contains the set stabilizer of a plane.
(iii) $\mathscr{D}$ is $AG(4, 2)$, G contains the translation group, and $G_x \simeq A_7$.

THEOREM 2. Let $\mathscr{D}$ be a $2-(v, K, 1)$ design, with K the set of odd integers. Suppose $G \leqslant \mathrm{Aut}\, \mathscr{D}$, and for all points $x \neq y$, G_{xy} fixes the line xy pointwise and has even order. Suppose further that no involution fixes three non-collinear points. Then one of the following holds.

(i) $0(G)_L$ is transitive on L for each line L. (In particular, $0(G)$ is transitive on points.)
(ii) $\mathscr{D}$ is $PG(2, 2^e)$, $e \geqslant 1$, and G fixes a line L, contains the translation group with respect to L, is solvable, and is flag-transitive on $AG(2, 2^e)$.
(iii) $\mathscr{D}$ is $PG(2, 2)$ and G is $PSL(3, 2)$.
(iv) $\mathscr{D}$ is $PG(3, 2)$ and $G \simeq A_7$.

Here $0(G)$ denotes, as usual, the largest normal subgroup of G having odd order. It is straightforward to deduce that the corresponding result for $4-(v, K, 1)$ designs is vacuous, where K is the set of odd integers.

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The above results are similar to Theorem 2 and the Corollary to Theorem 1 of [16]. In addition to dealing with $2-(v, K, 1)$ designs with $|K| \neq 1$, Theorem 2 removes the hypotheses in [16] that no nontrivial element of G fixes three non-collinear points, and that G_{xL}^{L} has at most one involution for $x \in L$ (cf. Section 5). The conclusions of Theorem 1 coincide with ones in [16]; those of Theorem 2 are less precise than ones found in [16]. There are, unfortunately, too many examples of Theorem 2 (i) to permit classification (Section 5), unless additional hypotheses are made (cf. Theorem 3.7). We have considered $2-(v, K, 1)$ designs, instead of 2-designs, since the more general situation is needed in [6]; while this generality creates difficulties in some parts of the proof, in one case (Proposition 3.3) it greatly simplifies matters.

Theorem 2 is proved in Section 3; Theorem 1 is then easily deduced (in Section 4) using some deep results on 2-transitive groups. While some of our arguments resemble ones in [16], the proof of Theorem 1 has been presented without reference to [16]. Most of the proof concerns the case $0(G) \neq 1$, the situation $0(G)=1$ being readily handled using the subgroup structure of the groups characterized in [1].

The crucial hypotheses of Theorem 2 are that K consists of odd integers, G_{xy} fixes xy pointwise, and G_{xy} has an involution fixing no point outside of xy (for all points x and y). Partial results exist in this more general setting; we hope to return to this at a later time.

2. PRELIMINARIES

Throughout this paper, X will denote a set (usually the set of points of a $t-(v, K, 1)$ design), and G a group of permutations of X. For $x, y \in X$, G_x is the stabilizer of x, and $G_{xy}=(G_x)_y$. For $\Delta \subseteq X$, G_Δ and $G(\Delta)$ are the set and pointwise stabilizers of Δ, and $G_\Delta^\Delta \cong G_\Delta/G(\Delta)$ is the group induced by G_Δ on Δ. For $S \subseteq G$, $\Omega(S)$ denotes the set of fixed points of S, and S^G is the conjugacy class of S in G.

If $\mathscr{D}$ is a $2-(v, K, 1)$ design and x, y are distinct points, xy denotes the unique line (i.e., block) through x and y. The symbol G_{xy} will, however, always refer to $G_x \cap G_y$.

$Z^*(G)$ is the subgroup of G such that $Z^*(G) \geqslant 0(G)$ and $Z^*(G)/0(G) = Z(G/0(G))$. Also, $G\#$ denotes $G-\{1\}$.

LEMMA 2.1. Let $A \lhd G$, $A \leqslant 0(G)$, and $\Delta = x^A$.

(i) If $t \in G_x$ is an involution, then $C_A(t)$ is transitive on $\Delta \cap \Omega(t)$.

(ii) If K is a Klein group fixing only one point $x \in \Delta$, then

$$|\Delta| = \prod_{t \in K\#} |\Delta \cap \Omega(t)|.$$

(iii) If $K \leqslant G_x$ is a Klein group, and $K^\Delta \neq 1$, then the three sets $\Delta \cap \Omega(t)$, $t \in K\#$, cannot coincide.

Proof. (i) Since $\langle t\rangle$ is Sylow in $\langle t\rangle A$, and $\langle t\rangle A$ is transitive on Δ, necessarily $N_{\langle t\rangle A}(\langle t\rangle)$ is transitive on $\Delta \cap \Omega(t)$.

(ii) By the Brauer-Wielandt theorem [20],

$$|A|\;|C_A(K)|^2 = \prod_{t\in K^{\#}} |C_A(t)|$$

and

$$|A_x|\;|C_{A_x}(K)|^2 = \prod_{t\in K^{\#}} |C_{A_x}(t)|.$$

By hypothesis, $C_A(K)$ fixes x. Hence,

$$|\Delta| = |A : A_x| = \prod_{t\in K^{\#}} |C_A(t) : C_{A_x}(t)| = \prod_{t\in K^{\#}} |\Delta \cap \Omega(t)|$$

by (i).

(iii) If these coincide, they are fixed by $A = \langle C_A(t) | t \in K^{\#}\rangle$. Since A is transitive, all three sets must contain Δ, so $K^{\Delta} = 1$.

A *subspace* of a $t-(v, K, 1)$ design is a set Δ of points such that, for each t-set $T \subseteq \Delta$, the block containing T is contained in Δ. If $|\Delta| > t$, Δ inherits a natural structure as a $t-(|\Delta|, K, 1)$ design, which is also denoted by Δ.

Lemma 2.2. Let $\mathscr{D}$ be a $2-(v, K, 1)$ design, with K a set of odd numbers. Let $t \in \mathrm{Aut}\,\mathscr{D}$ be an involution. Then $\Delta^t = \Delta$ for every subspace $\Delta \supseteq \Omega(t)$ of $\mathscr{D}$.

Proof. Let $x \in \Delta - \Omega(t)$. Then t fixes xx^t, where $|xx^t|$ is odd. Hence, t fixes some $y \in xx^t$. Thus, $y \in \Omega(t) \subseteq \Delta$ implies that $x^t \in xx^t = xy \subseteq \Delta$.

3. PROOF OF THEOREM 2

Let $\mathscr{D}$ and G be as in Theorem 2. Throughout our proof, t, u, and z will always denote involutions, x and y will always be points, and L will always be a line.

The lines through x yield a partition of $X - \{x\}$ into sets of even size. Hence, $v = |X|$ is odd.

By (2.1 i), it suffices to prove that either $\mathscr{D}$ is $PG(2, 2^e)$ or $PG(3, 2)$, or $0(G)$ is transitive. This will be proved by induction on v.

We may assume that G *is generated by its involutions*. Since each line is the set of fixed points of an involution, G moves each point.

Lemma 3.1. Let $S \subseteq G$.

(i) $\Omega(S)$ is a subspace.

(ii) If Σ is a subspace contained in no line, then G_{Σ}^{Σ} satisfies the conditions of the theorem.

Proof. (i) If $x, y \in \Omega(S)$ and $x \neq y$, then $S \subseteq G_{xy}$ fixes xy pointwise. Thus, $xy \subseteq \Omega(S)$.

(ii) Let L be a line, $L \subset \Sigma$. Let $t \in G(L)$. Then $L = \Omega(t)$, so $\Sigma^t = \Sigma$ by (2.2).

Lemma 3.2. Let $t \in G$ be an involution such that $\Omega(t)$ is a line L. Then the following hold.

(i) If G_L moves each point of L, then G_L^L has an orbit on which it acts faithfully as a Frobenius group having a complement of even order.

(ii) If G_L moves each point of L, then for $x \in L$, G_{xL}^L has at most one involution.

(iii) $\{x \in L | C(t)_x^L \neq 1\}$ is an orbit of $C(t)$.

(iv) Each point of $X - L$ is on a unique fixed line of t, and each fixed line meets L.

(v) If t fixes $L' \neq L$, then t centralizes an involution in $G(L')$.

Proof. Let $y \in X - L$. Then t fixes $yy^t = L'$, where $|L'|$ is odd, so t fixes a point of L'. This proves (iv). Moreover, t normalizes $G(L')$, and hence centralizes some involution $u \in G(L')$, so (v) holds. Clearly, $|u^L| = 2$. If $x, x' \in L$, $x \neq x'$, then by hypothesis $G_{xx'} = G(L)$. This implies (i)–(iii).

Proposition 3.3. If G_L^L is intransitive for some line L, then $\mathscr{D}$ is $PG(2, 2^e)$, $e \geqslant 1$, and G fixes a line $L^\#$, contains the translation group with respect to $L^\#$, is solvable, and has order $2^{2e+1}(2^e+1)$.

Proof. Choose $x \in L$ with $|G_{xL}^L|$ odd, and set $\Delta = x^G$. If an involution $t \in G_x$ fixes L, by (3.2 v) $t \in G(L)$, so t fixes only one line L on x. Then G_x moves L, and G_{xL} is strongly embedded in G_x. If now u is any involution in G_x, it must fix some line in L^{G_x}, and hence $\Omega(u) \in L^{G_x}$. This shows that G_x is transitive on the lines through x. Since $|\Delta| > 1$, it follows that Δ is contained in no line, and G is transitive on the lines meeting Δ. Then Δ, together with these lines, forms a design $\mathscr{D}^*$ with $v^* = |\Delta|$ and $k^* = |L \cap \Delta|$. Since G is transitive on the lines of $\mathscr{D}^*$, it is transitive on the points ([7], p. 78), and hence flag-transitive. Consequently, G^Δ is primitive (Higman-McLaughlin [15]). Since each line meeting Δ is in $\mathscr{D}^*$, each point of $\mathscr{D}$ is an intersection of lines of $\mathscr{D}^*$, so $G \cong G^\Delta$.

By (3.2 v), there is an involution in G_L^L, which then fixes no point of $L \cap \Delta$. Thus, k^* is even. We have seen $r^* = |G_x : G_{xL}|$ is odd. Hence, $v^* = (1 + r^*(k^* - 1))$ is even. In view of the primitivity of G, it follows that $0(G) = 1$.

Each line meets Δ in an even number of points (namely, 0 or k^*). Hence, $X - \Delta$ is either a point, or is contained in a line, or else inherits a natural structure as a $2-(v - v^*, K, 1)$ design $\mathscr{D}^\#$ (where K is as in Theorem 2).

Since G fixes no point, $X - \Delta$ is not a point. Suppose $X - \Delta \subseteq L^\#$ for some line L^*. Then $G(L^*) \lhd G$, so $G(L^*)$ is transitive on Δ. In particular, $L^\# \cap \Delta = \emptyset$. Since each involution fixes $L^\#$, each line meets $L^\#$ (by (3.2 iv)), so $r^* = |L^\#|$. By (3.2 v), $G_{L^\#}$ is transitive on L^*. Since G is generated by its involutions, by (3.2 i) $|G_{L^\#}^{L^\#}| = 2|L^\#|$. Then $|G(L)| = 2$, and G has a normal subgroup H of index 2 such that $H(L) = 1$. Here, H is still flag-transitive on $\mathscr{D}^*$, and $H_{xy} = 1$ for $x, y \in \Delta$, $x \neq y$. Thus, $H \simeq H^\Delta$ is a primitive Frobenius group. Let K be the Frobenius kernel of H, so $|K| = v^*$ is even. Then K is an elementary abelian 2-group, and hence $K \leqslant G(L^*)$. Moreover, $C_K(t)$ is transitive on $L - L^* \cap L$. Write $v^* = 2^e$ and $k^* = 2^f$, so $2^f - 1 | 2^e - 1$ and hence $f | e$. On the other hand, t induces an involutory linear transformation of K (regarded as a $GF(2)$-space), and hence fixes at least $\sqrt{|K|} = \sqrt{v^*}$ vectors. Thus, $f \geqslant e/2$. This shows that $\mathscr{D}^*$ is a translation plane of order k^*. That it is desarguesian is easy to check (alternatively, see Foulser [9]).

Now suppose $X - \Delta$ is the set of points of a $2-(v-v^*, K, 1)$ design $\mathscr{D}^\#$. Then clearly $|G(X-\Delta)|$ is odd, and hence $G(X-\Delta) = 1$ since $0(G) = 1$. For the same reason, $\mathscr{D}^\#$ must be $PG(2, 2^e)$ or $PG(3, 2)$.

Suppose $\mathscr{D}^\#$ is $PG(2, 2^e)$, and that G fixes a line $L^\#$ of $\mathscr{D}^\#$. Then we have just seen that $G(L^\#)$ has an elementary abelian 2-subgroup $K \lhd G$; it must be regular on Δ by primitivity. Also, $t \in G(L)$ fixes $L^\#$, so L meets $L^\#$. Proceeding as before, we find that $\mathscr{D}^*$ is an affine plane of order $k^* = \sqrt{|K|}$. $\mathscr{D}^\#$ has the same order. However, x is on $k^* + 1$ lines, at least one of which meets $X - (L^\# \cup \Delta)$; hence, all do, and in the same number l of points. It follows that $2^{2e} = l(k^* + 1) = l(2^e + 1)$, which is absurd.

Thus, $\mathscr{D}^\#$ is $PG(2, 2)$ and G is $PSL(3, 2)$, or $\mathscr{D}^\#$ is $PG(3, 2)$ and G is A_7. In either case, all involutions are conjugate, so $G(L)$ contains a normal Klein group. Since G_x is maximal in G, and has a strongly embedded subgroup $G(L)$, this is impossible.

REMARK. The preceding inductive proof should be compared with the more painful argument used in the corresponding part of [16] (Lemma 5.2).

PROPOSITION 3.4. If $|\Omega(t)| = 1$ for some involution z, then $0(\langle t^G \rangle)$ is transitive on X (and hence so is $0(G)$).

PROOF. Let $\Omega(z) = \{x\}$. By (2.2), z fixes every line on x. By (3.3), G_L is transitive for each L. Hence, by (3.2 i, iv), z is the only involution fixing just x. Thus, $z \in Z(G_x)$.

If $z' \in z^G$ commutes with z, it fixes x, and hence equals z. Thus, $z \in Z^*(G)$ (Glauberman [10]). By (3.3), it follows that $\langle z^G \rangle = \langle z \rangle 0(\langle z^G \rangle)$ is transitive on X.

PROPOSITION 3.5. Suppose $Z^*(G) > 0(G)$, let U_0 be a Sylow 2-subgroup of $Z^*(G)$ and U the group generated by the involutions in U_0. Then $0(\langle U^G \rangle)$ is transitive on X (and in particular, so is $0(G)$).

PROOF. Set $A=0(\langle U^G\rangle)$, so $\langle U^G\rangle=UA$ and $UA/A\leqslant Z(G/A)$. By (3.3), G is transitive on X, so each G_x contains a conjugate of U. By (3.4), we may assume $\Omega(z)$ is a line for each $z\in U^\#$. Fix such a z, and set $L=\Omega(z)$. By (3.3), $C(z)$ is transitive on L.

By (3.2 v), $C(z)$ contains an involution $t\notin G(L)$. Then $\Omega(t)\notin L^G$ (as otherwise, z would centralize a conjugate of itself lying in $G(\Omega(t))$). For each $x\in\Omega(t)$, $C(t)_x\cap z^G\neq\emptyset$, so $\langle z^G\rangle\cap C(t)$ is transitive on $\Omega(t)$. Similarly, $\langle z^G\rangle\cap C(tz)$ is transitive on $\Omega(tz)$.

Let $\langle t, z\rangle\leqslant G_x$, and set $\Delta=x^A$. Then Δ contains the lines $\Omega(t)$ and $\Omega(tz)$. By (2.1 iii), $G(L)^\Delta$ contains no Klein group. Consequently, if $|U|>2$ we may assume $t\in U$, and then $\langle t^G\rangle\cap C(z)$ will be transitive on $\Omega(z)$; thus, (3.5) holds in this case, so we may assume $U=\langle z\rangle$.

If now u is any involution in $C(z)-\{z\}$, then u commutes with some conjugate $u'\neq u$ of itself (Glauberman [10]). Since G has 2-rank 2 by (3.2 i), necessarily $u'u\in z^G$. Now $|\Delta|=|\Omega(u)|\,|\Omega(u')|\,|\Delta\cap L|$ by (2.1 ii), so $|\Omega(u)|=k$ is independent of u. Set $l=|\Delta\cap L$, so $|\Delta|=k^2l$.

G acts on $\mathscr{S}=\Delta^G$ as a transitive group, with $z^{\mathscr{S}}$ inducing a central element fixing Δ. Thus, z fixes each member of $\mathscr{S}$. Since $|\Delta|$ is odd, z fixes a point of each member, so $L=\Omega(z)$ meets each member of $\mathscr{S}$. Since $C(z)$ is transitive on L, $l=|L'\cap\Delta'|$ is independent of $L'\in L^G$ and $\Delta'\in\mathscr{S}$.

Now suppose $\Delta\neq X$, and let $\Delta'\in\mathscr{S}-\{\Delta\}$. There are exactly $|\Delta'|/l=k^2$ members of L^G on x. As $z\in Z^*(G_x)$, $A_x<\langle z^{G_x}\rangle$ is transitive on $z^G\cap G_x$, and hence on these k^2 lines. Let p be a prime dividing k, and P a Sylow p-group of A_x. Then each orbit of P on Δ' has length $\geqslant k_p^2$ (where k_p is the p-share of k). Since P acts on $\Delta-\{x\}$, it fixes some $y\in\Delta-\{x\}$, and hence $P\leqslant G(xy)$. Clearly, $xy\cap\Delta'=\emptyset$, so $|xy|=k$. Now P acts on $\Delta-xy$, $k(kl-1)$ points. We can thus find $x'\in\Delta-xy$ with $P_{x'}\neq 1$. Set $\Sigma=\Omega(P_{x'})$.

By (3.1), G_Σ contains a Klein group, so we can find $z'\in G_\Sigma\cap z^G$. By induction, $0(G_\Sigma)$ is transitive on Σ. By (2.1 ii), $|\Sigma|=k^2|L|$.

However, L meets each member of $\mathscr{S}$ in l points, so $|\mathscr{S}|=|L|/l$. Then $v=(|L|/l)|\Delta|=|L|k^2=|\Sigma|$, which is ridiculous.

The proof of Theorem 1 will require the following variation on (3.5).

LEMMA 3.5′. Suppose $Z^*(G)$ contains a Klein group $U=\{1, z, t, tz\}$. Assume further that $|\Omega(t)|$ and $|\Omega(tz)|$ are not relatively prime. Then $0(\langle z^G\rangle)$ is transitive on X.

PROOF. Set $A=0(\langle z^G\rangle)$ and $\Delta=x^G$ (where $U\leqslant G_x$). As in the proof of (3.5), $\Omega(t)\cup\Omega(tz)\subseteq\Delta$ and $L=\Omega(z)$ meets each member of $\mathscr{S}=\Delta^G$. By (2.1 iii), $G(L)$ contains no Klein group, so by (3.2 i) G has 2-rank 2. It follows that $G=0(G)U$. (Recall that G is generated by its involutions.) Also, $|\Delta|=k_1k_2l$, where $l=|\Delta'\cap L'|$ is independent of $\Delta'\in\mathscr{S}$ and $L'\in L^G$, $k_1=|\Omega(t)|$, and $k_2=|\Omega(tz)|$.

G_x is again transitive on the $|\Delta|/l=k_1k_2$ lines through x. Let p be a prime dividing (k_1, k_2), and P a Sylow p-group of G_x. Then P moves all lines through x, $P \leqslant G(xy)$ for some $y \in \Delta - \{x\}$, and $xy \subset \Delta$. We may assume $|xy| = k_1$. Then P acts on $\Delta - xy$, $k_1(k_2 l - 1)$ points. The transitivity of G_x implies that $(k_1 k_2)_p$ divides $|P|$, where $(k_1 k_2)_p > (k_1)_p$. Hence, P cannot be semiregular on $\Delta - xy$. This leads to the same contradiction as in (3.5).

From now on we will assume that $Z^*(G) = 0(G)$.

LEMMA 3.6. One of the following holds.

(i) A Sylow 2-subgroup S of G is dihedral, quasidihedral, wreathed $Z_{2^m} \wr Z_2$, or $Z_{2^m} \times Z_{2^m}$ for some m.

(ii) G has a proper normal subgroup K, with $|G : K|$ a power of 2, such that the stabilizer K_L of some line L is a strongly embedded subgroup of K.

PROOF. ([16], (5.1)). Suppose (i) does not hold, and let $K = 0^{2'}(G)$. Let $t \in K$, so t is in every normal subgroup having index 2 in G. Hence, by Harada [12], Theorem 2, $t' \in (t^G - \{t\}) \cap C(t)$ implies that $\Omega(t) = \Omega(t')$.

Now fix $t \in Z(S \cap K)$, and set $L = \Omega(t)$. Then t cannot fix any $L' \in L^G - \{L\}$ (as it would then centralize some $t' \in t^G \cap G(L')$). If t^G consists of all involutions in K, this proves (ii). Let u^K be a class of involutions of K disjoint from t^G.

By [10], we can find $u, u' \in S \cap u^K$ with $uu' = u'u \neq 1$. We know $\Omega(u) = \Omega(u')$, so $\Omega(u) = \Omega(uu')$. Since $\langle u, u' \rangle$ acts on $\Omega(t) = L$, it follows that $\Omega(u) = L$. Now let $t' \in t^G$ with $\Omega(t') \neq L$. Then $\langle t', u \rangle$ contains an involution z. Since $\langle z, u \rangle$ fixes $\Omega(u) = L$, it centralizes some $t'' \in t^G \cap G(L)$, $\langle u, t'' \rangle \leqslant G(L)$ acts on $\Omega(z)$, and hence $\Omega(z) = L$. But now $\langle u, z \rangle \leqslant G(L)$ acts on $\Omega(t') = L' \neq L$, so this is a contradiction.

LEMMA 3.7. If $N \lhd G$, $N \leqslant 0(G)$, and N fixes a line, then $N = 1$.

PROOF. Suppose $N \neq 1$, and let N fix L. By (3.3), $\mathscr{L} = L^G$ is an imprimitivity system for G. In particular, G has at least two classes of involutions.

Suppose (3.6 i) holds. Then (since $Z^*(G) = 0(G)$) $K = 0^{2'}(G)$ has a single class of involutions. Since no involution in $G(L)$ can fix a line of $\mathscr{L} - \{L\}$, it follows that (3.6 ii) holds.

Note that $|G(\mathscr{L})|$ is odd. For otherwise, $Z^*(G) = 0(G)$ implies that $G(\mathscr{L})$ contains a Klein group K. Then K fixes a point of each line in $\mathscr{L}$, so $\Omega(K)$ is a line. Since K then acts faithfully on L, this contradicts (3.2 i).

Consequently, $G(\mathscr{L}) \leqslant 0(G)$. In particular, $Z^*(G^{\mathscr{L}}) = 0(G^{\mathscr{L}})$. By Bender [3], $0(G^{\mathscr{L}}) = 1$, so $G(\mathscr{L}) = 0(G)$. Moreover, $G^{\mathscr{L}}$ has a normal subgroup $H \cong PSL(2, 2^e)$, $Sz(2^e)$, or $PSU(3, 2^e)$ for some $e \geqslant 2$, acting on $\mathscr{L}$ as usual.

Let t be an involution with $\Omega(t) \notin \mathscr{L}$. Then t fixes $|\Omega(t)|$ members of $\mathscr{L}$. By (3.2 i), $C_{G(L)}(t)$ has no Klein group. Thus, by considering $t^{\mathscr{L}}$ and using standard properties of H, we find that $G^{\mathscr{L}}$ is $P\Gamma L(2, 4)$ and $|\mathscr{L}|=5$. In particular, if $|L|=k$ then $v=5k$. Moreover, $|\Omega(t)|=3$.

Let $L' \in \mathscr{L}$ and $x \in L \neq L'$. We claim that $W=G(L')_x$ is 1. For suppose $W \neq 1$. Then $\Delta=\Omega(W)$ is a subspace, and induction applies to G_Δ^Δ (by (3.1)). G_Δ^Δ is transitive. (For otherwise, Δ is $PG(2, 2)$, $k=3$, and $W \lessdot G_x$ fixes L, so $L \subset \Omega(U)$ misses L'.) Thus, $k \,||\Delta|$, so $|\Delta|=3k$. However, $W=G(L') \cap G(L)$ is normalized by $G_{LL'}$, where $G_{LL'}$ is transitive on $\mathscr{L}-\{L, L'\}$. This contradiction proves our claim.

Fix $L' \in \mathscr{L}-\{L\}$ and $x \in L$. For each $x' \in L'$, there is an involution in $G_{xx'L}$ fixing only one point of L'. Hence, $G_{xLL'}$ is transitive on L'. Moreover, $G_{xx'L}^{L'}$ is a Frobenius complement, so $G_{xL'}$ has a normal subgroup A with $A^{L'}$ regular. Here, $A \leqslant G(L)$. (For otherwise, A has a nontrivial p-subgroup P for some prime $p|k-1$, and then $P^{L'}=1$ implies that $P \leqslant G(L')_x=1$.) Thus, $|A|=k$, and $A=G(L)_{L'}$. Set $A'=G(L')_L$. Then $AA'=A \times A' \lhd G_{LL'}$.

Let t fix L and L'. Since A is faithful on L', by considering $(\langle t\rangle A)^{L'}$ we see that t inverts A. Similarly, t inverts A'. Thus, t inverts AA'.

We claim that AA' is semiregular on $X-(L \cup L')$. For suppose $(AA')_y \neq 1$ with $y \notin L \cup L'$. Let $y \in L'' \in \mathscr{L}$. Since $(|AA'|, k-1)=1$, $(AA')_y \leqslant G(L'')$. However, t inverts $(AA')_y$, and hence fixes L''. Since $t^{\mathscr{L}}$ is any involution in S_5 fixing L and L', this is a contradiction.

Thus, $k^2|3k$, so $k=3$ and $v=15$. It follows that $|(AA')^{\mathscr{L}}|=3$ and $|0(G)|=3$.

Let $G \rhd G^+ > 0(G)$ with $G^{+\mathscr{L}}=A_5$. Then $C_{G^+}(0(G))^{\mathscr{L}}=A_5$, so $G^+= =0(G) \times U$ with $U \cong A_5$. Set $\Sigma=x^U$. Then $|\Sigma|=5$, and U^Σ contains distinct involutions of the form $t_i=(x, x')(y_i) \ldots$, $i=1, 2$. Both fix xx', so $|xx'| \geqslant 4$. This contradiction proves (3.7).

THEOREM 3.8. Suppose $Z^*(G)=0(G) \neq 1$. Then $\mathscr{D}$ is an affine space $AG(3, k)$, and $G=SL(3, k)T$ with T the translation group. (In particular, $0(G)$ is transitive on X.)

PROOF. Let N be any nontrivial normal subgroup of G with $N \leqslant 0(G)$. Let $x \in X$, and consider $\Delta=x^N$.

Since G is transitive, N fixes no point. By (3.7), Δ is contained in no line. By (2.1 iii) (applied to G_Δ^Δ), $G(L)$ does not contain a Klein group for each line L. By (3.2 i), G has 2-rank 2 (cf. (3.6)). Moreover, $C(t)^{\Omega(t)}$ is now transitive for each involution t.

Consequently, if z is an involution central in a Sylow 2-group S of G_x, then a Klein group $K \leqslant S$ exists having all its involutions conjugate to z in G. By the transitivity of $C(z)^{\Omega(z)}$, these involutions are all conjugate in G_x. Hence, by (2.1), $|\Delta|=k^3$, where $k=|\Delta \cap \Omega(z)|$.

Now let $t \notin z^G$ be any involution in S. Then t commutes with some

$t' \in S \cap (t^G - \{t\})$ (by [10]). Since G has 2-rank 2 and $\langle t, t'\rangle$ commutes with z, necessarily $tt' = z$. By (2.1 ii), $|\Delta| = k\ |\Delta \cap \Omega(t)|^2$, so $|\Delta \cap \Omega(t)| = k$.

Thus, each line meets Δ in k points. The lines meeting Δ turn Δ into a design $\mathscr{D}_\Delta$, inheriting the hypotheses of Theorem 2. Moreover, as $|G(\Delta)|$ is odd and G_x acts on Δ, the hypotheses of (3.8) are inherited. Finally, note that there are exactly $(|\Delta|-1)/(k-1) = k^2+k+1$ lines on x; this number is independent of N. Hence, $\Delta = x^{0(G)}$.

Case 1. $0(G)$ is transitive on X.

Choose N to be a minimal normal subgroup of G contained in $0(G)$. We have just seen that $x^N = x^{0(G)} = X$. Thus, N is regular on X, and can be regarded as a vector space over $GF(p)$ for some prime p. Identify X with N, via $x^n \equiv n$, $x \equiv 0$.

If $t \in G_x$ then $C_N(t)$ is regular on $\Omega(t)$ (by (2.1 i)). We can thus regard the lines through x as subspaces of N. The remaining lines are obtained by applying elements of N, and hence are just the translates of the lines through 0. The automorphism $\sigma = -1$ of N fixes each subspace of N, and hence $\sigma \in \operatorname{Aut} \mathscr{D}$. (Clearly, $\sigma \notin G$.)

We have $N = C_N(t) \oplus [N, t]$, where $[N, t] = \{n \in N | n^t = -n\} = \Omega(\sigma t)$ has order k^2 and is normalized by $C(t)$. If $x \in L = L^t \neq \Omega(t)$ then $(\sigma t)^L = 1$, so $L \subset \Omega(\sigma t)$. Since $[N, t]$ is transitive on $\Omega(\sigma t)$, it follows that $\Omega(\sigma t)$ is a subspace having k^2 points, i.e., an affine plane.

This provides us with a set of affine planes through 0, each of which is a subspace of both X and N. Consider two of these, say E_1 and E_2. Since these are subspaces of N, $|E_1 \cap E_2| \geqslant k$. There is a unique line through 0 and a point $\neq 0$ of $E_1 \cap E_2$, and this line must be in both E_1 and E_2 (as both are subspaces of X). Since $E_1 \cap E_2$ is certainly a subspace of each affine plane, clearly $E_1 \cap E_2$ must be contained in a line. Hence, any two planes on 0 meet in a line.

Let $\mathscr{P}(x)$ be the structure consisting of the lines and planes through x. Each such plane has $k+1$ such lines, and two such planes have a unique common line. There are $(v-1)/(k-1) = k^2+k+1$ lines.

We next show that each line L on x is in $k+1$ planes. Since the planes $\supset L$ induce a partition of $X - L$, there are at most $(v-k)/(k^2-k) = k+1$ such planes. Conversely, let $t \in G(L)$. Then $t^{\Omega(\sigma t)}$ fixes each line L' of $\Omega(\sigma t)$ on x, and centralizes an involution $u \in G(L')$. Consider $\Omega(\sigma tu) = [N, tu]$. This is fixed by $\langle t, u\rangle$, and tu induces a dilatation, so $\Omega(t) \cup \Omega(u) \subset \Omega(\sigma tu)$. However, $\Omega(u) \subseteq \Omega(\sigma t) \cap \Omega(\sigma tu)$, $\Omega(t) \subset \Omega(\sigma tu)$, and $\Omega(t) \not\subset \Omega(\sigma t)$. Thus, $\Omega(\sigma t) \cap \Omega(\sigma tu) = \Omega(u) = L'$. This means that each such line L' is in a plane containing L. There are $k+1$ choices for L', and these produce $k+1$ planes containing L.

Thus, the dual of $\mathscr{P}(x)$ is a $2-(k^2+k+1, k+1, 1)$ design. This proves that $\mathscr{P}(x)$ is a projective plane.

In particular, any two concurrent lines are in one of our affine planes. By Sasaki [19] or Buekenhout [5], $\mathscr{D}$ is $AG(3, k)$. Moreover, G_x induces a collineation group of the desarguesian projective plane $\mathscr{P}(x)$ such that each line is fixed pointwise by an involution. (Namely, $[N, t]$ is fixed pointwise by t, if it is regarded as a line of $\mathscr{P}(x)$.) Since $Z^*(G_x) = 0(G_x)$, it follows that G_x induces at least $PSL(3, k)$ on $\mathscr{P}(x)$ (see, e.g., [7], p. 196). We are assuming that G is generated by its involutions. This proves (3.7) in this case.

CASE 2. $0(G)$ is intransitive on X.

Once again, $\Delta = x^N = x^{0(G)}$. We have seen that $\mathscr{D}_\Delta$ and G_Δ^Δ inherit all the hypotheses of (3.8). Thus, by Case 1, $\mathscr{D}_\Delta$ is $AG(3, k)$ and $G_\Delta^\Delta \trianglerighteq SL(3, k) \cdot T$.

Now $G_{x\Delta}$ is transitive on the lines through x. Since G is transitive on X (by (3.3)), it follows that $\mathscr{D}$ is a design and G is flag-transitive on $\mathscr{D}$. Hence, G is primitive on X (Higman-McLaughlin [15]). However, this contradicts the fact that $0(G)$ is intransitive on X.

PROPOSITION 3.9. If $0(G) = 1$, then $\mathscr{D}$ is $PG(2, 2)$ or $PG(3, 2)$.

PROOF. By (3.3), we may assume that G_L^L is transitive for all L. Then G is transitive, so $0_2(G) = 1$ as $|X|$ is odd.

Suppose first that (3.6 ii) holds. Since $0(K) = 1$, K acts on $\mathscr{L} = L^K$ as $PSL(2, 2^e)$, $Sz(2^e)$, or $PSU(3, 2^e)$, for some $e \geqslant 2$, in its usual 2-transitive representation (Bender [3]). Let $t \in (G - K) \cap S$ for a Sylow 2-group S of G. The proof of (3.6) shows that $S \cap K \leqslant K(L')$ for some $L' \in \mathscr{L}$. We have $G = SK$, and hence $G \leqslant \text{Aut}\,(K)$. Thus, K is not $Sz(2^e)$; moreover $|C_{Z(S \cap K)}(t)|$ is $2^{e/2}$ if $K \cong PSL(2, 2^e)$ and is 2^e if $K \cong PSU(3, 2^e)$. We, may assume that $\Omega(t) \notin \mathscr{L}$. Then by (3.2 i), the elementary abelian group $C_{Z(S \cap K)}(t)$ has order 2. Thus, K is $PSL(2, 4)$, so $G \cong S_5$. $\mathscr{D}$ has $5 + 10$ lines, while $|G : G_x| = v$ is odd. This easily yields a contradiction.

Let M be a minimal normal subgroup of G. Then by (3.6 i), Brauer [4], and Alperin-Brauer-Gorenstein [1], $M \cong PSL(2, q)$, $PSL(3, q)$, $PSU(3, q)$, A_7, or M_{11}, for some odd q. In particular, M has a single class of involutions.

If t is any involution, then certainly $C(t) \leqslant G_{\Omega(t)}$. In particular, if $C_M(t)$ is a maximal subgroup of M then $C_M(t) = M_{\Omega(t)}$ and $C_M(t)$ has a homomorphic image as in (3.2 i). Similarly, if $C_G(t) \leqslant H < G$ always implies that $t \in Z^*(H)$, then $t \in Z^*(G_{\Omega(t)})$, so $C(t)$ has $G_{\Omega(t)}^{\Omega(t)}$ has a homomorphic image.

This eliminates all but the following cases: M is A_7, M_{11}, $PSL(3, 3)$, $PSU(3, 3)$, or $PSL(2, q)$. Moreover, these properties of $C(t)$ show that $G = M$ in the first two cases, that $\mathscr{D}$ is a $2-(v, 3, 1)$ design in the first four, and that G is $PSL(2, q)$ or $PGL(2, q)$ in the last case.

Suppose $M(L)$ contains a Klein group $\langle z, z_1 \rangle$, for some line L. Then $\langle C_M(z), C_M(z_1) \rangle \leqslant M_L < M$. It follows that M is A_7, $PSL(2, 7)$, $PSL(2, 5)$, or $PSL(2, 9)$ (Dickson [8]). The first two cases lead to $PG(3, 2)$ and $PG(2, 2)$. The last two cannot occur.

Thus, for $z \in M$, we may assume $z \in Z^*(G_{\Omega(z)})$. Then $C(z)$ is transitive on $\Omega(z)$ (by (3.3)), and hence G_x is transitive on $z^G \cap G_x$ if $z \in G_x$.

Now suppose M is A_7, M_{11}, $PSL(3, 3)$, or $PSU(3, 3)$. Since all involutions fix just 3 points, $|G : G_x| = v \equiv 3 \pmod 4$. Also, M_x has a single class of involutions, and contains a Sylow 2-group of M. Clearly, G is not 4-transitive on X. Consequently, these cases cannot arise.

This leaves us with the possibilities $G \simeq PSL(2, q)$ or $PGL(2, q)$. Let $z \in M$, and let $t \in G - M$ if $G \neq M$. Suppose $q \equiv \varepsilon \pmod 4$, where $\varepsilon = \pm 1$. Then $|\Omega(z)| = k$ divides $(q-\varepsilon)/2$ and $|\Omega(t)| = k'$ divides $(q+\varepsilon)/2$.

The case $G \simeq PGL(2, q)$ can be eliminated as follows. Each involution u is the unique involution in $G(\Omega(u))$, and $C(u)^{\Omega(u)}$ is transitive. By (3.2 v), $C(u)$ yields a partition of $X - \Omega(u)$ into sets of size $k-1$ and $k'-1$. Applied to $u = z$ or t, this implies that

$$v - k = \tfrac{1}{2}(q-\varepsilon)(k-1) + \tfrac{1}{2}(q-\varepsilon)(k'-1)$$

$$v - k' = \tfrac{1}{2}(q+\varepsilon)(k-1) + \tfrac{1}{2}(q+\varepsilon)(k'-1).$$

Then $k' - k = -\varepsilon(k-1+k'-1)$, which is absurd.

Thus, G is $PSL(2, q)$, all lines have k points, and $v - k = \frac{1}{2}(q-\varepsilon)(k-1)$. In particular, $r - 1 = \frac{1}{2}(q-\varepsilon)$; moreover, z fixes $(r-1)/k$ blocks through each point of $\Omega(z)$, so $k \mid (v-k)$. On the other hand, there are $vr/k = |G : C(t)| = \frac{1}{2}q(q+\varepsilon)$ lines. Thus, $r = 1 + \frac{1}{2}(q-\varepsilon)$ divides $\frac{1}{2}q(q+\varepsilon)$, so $\varepsilon = 1$. Now $k + \frac{1}{2}(q-1)(k-1) = v = qk$, so $(q-1)(k+1) = 0$, which is ridiculous.

This completes the proof of Theorem 2.

4. PROOF OF THEOREM 1

Let $\mathscr{D}$ and G be as in Theorem 1. For $x \in X$, let $\mathscr{D}_x$ consist of $X - \{x\}$, together with the blocks on x with x removed. Then Theorem 2 applies to $\mathscr{D}_x$ and G_x.

In cases (iii) and (iv), G_x is 2-transitive on $X - \{x\}$. Since G certainly moves x, G is 3-transitive, and the result follows readily.

Suppose (ii) holds for some x. Then G_x fixes some block B on x, and $G(B)$ has a normal elementary abelian 2-subgroup regular on $X - B$. Moreover, G_x is transitive on $B - \{x\}$. Once again, G moves x.

Consider the possibility that G is transitive on X. Here, the transitivity of $G(B)$ implies that $G_{x'}$ fixes B whenever $x' \in B$, so G_B^B is transitive. It follows that B^G is an imprimitivity system for G. However, this implies that $|B| = 2^e + 2$ divides $|X| = 2^{2e} + 2^e + 2$, so $e = 1$ and $\mathscr{D}$ is $AG(3, 2)$.

Thus, we may assume G is intransitive on X. We know $G(B)$ is transitive on $X - B$. For $y \in X - B$, we cannot have G_y transitive on $X - \{y\}$. Thus, (ii) must also hold for $\mathscr{D}_y$. Then G_y fixes some block C; clearly $B \cap C = \emptyset$. Then the transitivity of $G(C)$ readily implies that of G.

We are thus left with the possibility that (i) holds for every $x \in X$. Here, G is 2-transitive on X. For each block B, G_B^B is also 2-transitive; also, all its involutions fix at most two points, while some fix two (see (3.2 v)). Thus, for each B, $G_B^B \trianglerighteq PSL(2, q)$ for some odd q, or G_B^B is A_6

(Hering [13]). We may assume that no involution of G fixes exactly two points (Kantor-Seitz [17], Theorem D).

Let A be a minimal normal subgroup of G_x contained in $0(G_x)$. Then A is an elementary abelian p-group for some p. We may assume that A is intransitive on $X-\{x\}$ (Hering-Kantor-Seitz [14]), semiregular on $X-\{x\}$ (O'Nan [18]), and also that $C(A)$ is semiregular on $X-\{x\}$ (Aschbacher [2]).

Let $K \leqslant G_{xy}$ be a Klein group (see (3.2 v)). Let $t \in K^{\#}$, and suppose $C_A(t) \neq 1$. Set $B = \Omega(t)$. By (2.1 iii), $G(B)$ has no Klein group, so $G_B = G(B)C(t)$. Hence, $C(t)^B \trianglerighteq PSL(2, q)$ and $C(t)_{xy}$ is irreducible on $C_A(t)$. Thus, $C_A(t)$ is transitive on $B-\{x\}$.

If now $C_A(t) \neq 1$ for all $t \in K^{\#}$, then two applications of (2.1 ii) yield

$$|A| = \prod_{t \in K^{\#}} (|\Omega(t)|-1) = |y^{0(G_x)}| = |X|-1,$$

whereas we are assuming A is intransitive on $X-\{x\}$. Thus, $C_A(z)=1$ for some $z \in K^{\#}$. If $C_A(z')=1$ for some $z' \in K-\langle z\rangle$, then zz' centralizes A and fixes y, which we are assuming does not occur. Hence, if $K^{\#}=\{z, t, t'\}$, then (2.1) yields $|A| = (|\Omega(t)|-1)(|\Omega(t')|-1)$ and $|X|-1 = |A|/(|\Omega(z)|-1)$.

$C(A)$ is semiregular on $X-\{x\}$; in particular it has odd order. If $C(A) > A$, we can apply the preceding argument with $C(A)$ in place of A, and deduce that $C(A)$ is regular on $X-\{x\}$. We may thus assume that $C(A) = A$. Then $\langle z\rangle A \lhd G_x$.

Now (3.5) and (3.5') imply that $A = 0(\langle z^{G_x}\rangle)$ is transitive on $X-\{x\}$. This contradiction completes the proof of Theorem 1.

5. CONCLUDING REMARKS

The following consequence of Theorem 2 is a slight strengthening of [16], Theorem 2.

COROLLARY. Let $\mathscr{D}$ be a $2-(v, k, 1)$ design with k odd, and $G \leqslant \mathrm{Aut}\, \mathscr{D}$. Suppose that, for any distinct points x and y, G_{xy} fixes the line xy pointwise, has even order, and is semiregular off xy. Then $\mathscr{D}$ is $PG(2, 2)$, $PG(3, 2)$, or an affine translation plane.

PROOF. By Theorem 2, we may assume that G_L^L is transitive for each line L. Then, for $x \in L$, G_{xL}^L is a Frobenius complement, and hence has a unique involution. Consequently, the corollary follows from [16], Theorem 2.

We next present some examples which indicate the difficulties involved in obtaining a complete classification of all the occurrences of Theorem 2(i).

EXAMPLE 1. Let $\mathscr{P}$ be an affine semifield plane of odd order k (a desarguesian plane will suffice). Adjoin its line at infinity L_∞. Let T be the translation group, and U a group of k elations with center $p_\infty \in L_\infty$ and affine axis L. Then $\mathscr{P}$ admits an involutory (a, A)-homology whenever $a = p_\infty \notin A$ or $a \in L_\infty - \{p_\infty\}$ and $p_\infty \in A$. Thus, $\mathscr{P}$ satisfies the hypotheses of Theorem 2.

We now diminish $\mathscr{P}$ as follows. Let G be the group generated by the aforementioned involutions. Then $G = (TU)K$, with K a Klein group normalizing both T and U. (Thus, K has an involution with axis L.) Note that $Z = [T, U]$ is the group of translations with center p_∞. Hence, U centralizes T/Z. Let T^* be any proper subgroup of T, containing Z properly, and normalized by K. (Such a T^* will exist provided k is not prime.) Set $X = x^{T^*}$, where $x \in L$, and let $\mathscr{D}$ have point set X and lines the intersections of size >1 of lines of $\mathscr{P}$ with X.

We claim that $\mathscr{D}$ meets our requirements. In fact, by construction $T^* \lhd G$, so G induces a group on $\mathscr{D}$ (namely, $G_x T^*$). The lines of $\mathscr{D}$ through x are the lines of $\mathscr{P}$ through x, and each is fixed pointwise by an involution in G_x. This proves our claim.

Note, however, that lines do not all have the same size. Moreover, if $|T^* : Z| = p$ is prime, most lines will have size p. (To see that T^* can be chosen this way, let $K = \langle z, t \rangle$ with z a dilation and t a homology having center p_∞. Then z inverts T, while $[T, t] = Z$, so K normalizes every subgroup of T/Z.)

Example 2. Let $\mathscr{A}$ be $AG(3, k)$, where k is odd and not a prime. Let T be its translation group. Single out a line L and a plane $E > L$. There is a group $U < SL(3, k)$ of order k^3 fixing L, E, and a point $x \in L$. Here, U centralizes T/T_E. Choose $T > T^* > T_E$ normalized by K, a Klein group in $SL(3, k)$ normalizing U. Now proceed as in Example 1.

Note that Examples 1 and 2 are, respectively, instances of (3.4) and (3.5′).

Example 3. It seems likely that examples exist which are designs having $v = k^3$ and $r = k^2 + k + 1$, and which are not affine spaces. For example, I believe examples will exist having the following form. G has a regular normal subgroup N of order k^3. Also, $G = NHK$, with H a group of order k^3, K a Klein group normalizing H, and $HK = G_x$. The lines through x have a natural structure as a semifield plane $\mathscr{P}(x)$, with the group HK playing the role of G in the first paragraph of Example 1. G_x fixes a unique plane E on x (corresponding to the line at infinity of $\mathscr{P}(x)$). E is itself a semifield plane, and G_E^E is again as in Example 1.

If N is elementary abelian, it is not hard to show (as in (3.7)) that $AG(3, k)$ is the only design of the above sort. However, it seems quite plausible that such a $\mathscr{D}$ exists with N nonabelian.

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