JOURNAL OF COMBINATORIAL THEORY, Series A **58**, 153–157 (1991)

Note

Generalized Quadrangles, Flocks, and BLT Sets

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Communicated by the Managing Editors

Received December 15, 1989

Approximately five years ago Thas observed a remarkable coincidence relating certain generalized quadrangles constructed in [Ka3] to flocks of cones and translation planes [Th] (cf. [Ka2]). While there is a rapidly growing literature concerning this connection, there as yet has been no explanation for it. This note contains an observation providing at least some kind of explanation. It also contains a seemingly useless method for "embedding" translation planes associated with the flocks "into" the generalized quadrangle. Just as in [BLT] we will only be able to deal with the case of odd parameters.

It is not the purpose of this note to introduce all of the notation required. Instead, we refer to [Ka2; Ka3; Pa1; Pa2; Th]. Let $F = GF(q)$ and $Q = F^2 \times F \times F^2$, with multiplication in Q given by $(u, c, v)(u', c', v') = (u+u', c+c'+v \cdot u', v+v')$, where $v \cdot u'$ denotes the dot product. The construction in [Ka3; Pa1; Pa2] of generalized quadrangles with parameters $s = q^2$, $t = q$, assumes that 2×2 matrices B_r and $M_r = B_r + B_r^t$, $r \in F$, are given, and the construction uses the following family $\mathscr{F}$ of $q+1$ subgroups of Q:

$$A(\infty) = 0 \times 0 \times F^2 \qquad \text{and} \qquad A(r) = \{(u, uB_r u^t, uM_r) \mid u \in F^2\}, r \in F,$$

each of order q^2. If also $M_r = C_r + C_r^t$ then $uB_r u^t = uC_r u^t$. Consequently, since we will assume that q is odd, without loss of generality $M_r = 2B_r$. We refer to the above papers for the exact construction of a generalized quadrangle $\mathbf{Q}(\mathscr{F})$ on which Q acts faithfully. In that construction, a necessary and sufficient condition on the matrices M_r is that $-\det(M_r - M_s)$ is a nonsquare in F for all distinct $r, s \in F$.

* Research supported in part by NSF Grant DMS 87-01794 and NSA Grant MDA 904-88-H-2040.

0097-3165/91 $3.00

The group Q has the following properties (cf. [Ka1, esp. p. 217]): $Z(Q) = 0 \times F \times 0$; $\bar{Q} := Q/Z(Q)$ is elementary abelian of order q^4; the group $Z^* \cong F^*$ of automorphisms $(u, c, v) \to (\alpha u, \alpha^2 c, \alpha v)$, $\alpha \in F^*$, of Q induces a subgroup of $\mathrm{Aut}(\bar{Q})$ that acts as a group of scalar transformations turning $\bar{Q}$ into a four-dimensional space over F; and the commutator $[gZ(Q), hZ(Q)]$, $g, h \in Q$, defines a nonsingular alternating bilinear form on that vector space when $Z(Q)$ is identified with F. Note that Z^* normalizes every member of $\mathscr{F}$.

The groups $\overline{A(r)} = A(r)\,Z(Q)/Z(Q)$, $r \in F \cup \{\infty\}$, are totally isotropic 2-spaces—i.e., lines—of this symplectic space. Write $\bar{\mathscr{F}} = \{\overline{A(r)} \mid r \in F \cup \{\infty\}\}$.

Lemma. *There do not exist three nonzero vectors of $\bar{Q}$ that lie in three different lines $\overline{A(r)}$ and are pairwise perpendicular (i.e., that lie in a totally singular line).*

Proof. Let r and s be distinct elements of F, and let $u, v \in F^2 - \{0\}$. Assume that (u, uM_r) and (v, vM_s) are perpendicular; this occurs if and only if $u \cdot vM_s = v \cdot uM_r$, that is, if and only if $u(M_r - M_s)\,v^t = 0$. Since $-\det(M_r - M_s)$ is a nonsquare, $u(M_r - M_s)\,u^t \neq 0$ (cf. [Ka3]), so that v cannot be a scalar multiple of u and hence the lemma holds for the three lines $\overline{A(r)}$, $\overline{A(s)}$, and $\overline{A(\infty)} = 0 \times F^2$. Next, let $k \in F - \{r, s\}$, and assume that $0 \neq (w, wM_k) \in \langle (u, uM_r), (v, vM_s) \rangle$. We may assume that $(w, wM_k) = (u, uM_r) + (v, vM_s)$, in which case $(u + v)\,M_k = uM_r + vM_s$ and hence $u(M_r - M_s)(M_r - M_k)(M_k - M_s)^{-1}\,u^t = 0$, which is impossible since $u \neq 0$ and $-\det(M_r - M_s)(M_r - M_k)(M_k - M_s)^{-1}$ is a nonsquare. ∎

Now apply the Klein correspondence. Totally isotropic lines of $\bar{Q}$ are sent to singular points of a certain five-dimensional orthogonal space V (see below for a coordinate description). In particular, $\bar{\mathscr{F}}$ is sent to a set $\mathrm{S}(\mathscr{F})$ of $q+1$ singular points. Moreover, the lemma states that any three members of $\mathrm{S}(\mathscr{F})$ have the property that *there is no singular point perpendicular to all three of them.* Such a set of $q+1$ points of V will be called a *BLT set,* since these sets were first introduced in [BLT].

We will require the following groups: the group $\Gamma Sp(\bar{Q})$ of all semilinear transformations projectively preserving the form on $\bar{Q}$; the corresponding projective group $P\Gamma Sp(\bar{Q})$; the group $\Gamma O(V)$ of all semilinear transformations projectively preserving the form on V; and the corresponding projective group $P\Gamma O(V)$. The Klein correspondence induces the familiar isomorphism $P\Gamma Sp(\bar{Q}) \cong P\Gamma O(V)$. Note that $\mathrm{Aut}(Q) = \bar{Q} \rtimes \Gamma Sp(\bar{Q})$, where the subgroup $\Gamma Sp(\bar{Q})$ of $\mathrm{Aut}(Q)$ is chosen so as to centralize Z^*, so that $P\Gamma Sp(\bar{Q}) = \Gamma Sp(\bar{Q})/Z^*$. There are $|\bar{Q}|$ conjugates of Z^*, and hence also $|\bar{Q}|$ subgroups of $\mathrm{Aut}(Q)$ conjugate to this group $\Gamma Sp(\bar{Q})$, but only one of these is important in the present context: there is only one conjugate of Z^* normalizing every member of $\mathscr{F}$.

THEOREM. *Let q be odd. Then the mapping $\mathscr{F} \to \mathrm{S}(\mathscr{F})$ has the following properties.*

(i) *Every BLT set arises as $\mathrm{S}(\mathscr{F})$ for a unique family $\mathscr{F}$.*

(ii) *If two families $\mathscr{F}$ and $\mathscr{F}'$ are equivalent under $\Gamma Sp(\bar{Q})$ then the corresponding BLT sets $\mathrm{S}(\mathscr{F})$ and $\mathrm{S}(\mathscr{F}')$ are equivalent under $\Gamma O(V)$.*

(iii) *The set-stabilizers $P\Gamma Sp(\bar{Q})_{\mathscr{F}}$ and $P\Gamma O(V)_{S(\mathscr{F})}$ are isomorphic, with an isomorphism induced by the Klein correspondence.*

Proof. Assertions (ii) and (iii) are immediate consequences of the Klein correspondence. Part of (i) is essentially contained in [BLT], but for completeness will be proved again here.

The Klein correspondence relates the four-dimensional vector space $\bar{Q}$ to the six-dimensional space $\bar{Q} \wedge \bar{Q}$ equipped with a quadratic form of Witt index 3. The 2-space $\langle(1, 0, c, -d), (0, 1, a, b)\rangle$ corresponds to the singular point $\langle(1, a, b, c, d, -ad-bc)\rangle$, the quadratic form being $(x_1, x_2, x_3, x_4, x_5, x_6) \to x_1x_6 + x_2x_5 + x_3x_6$. We may assume that (for $r \in F$) $2B_r = M_r = \left(\begin{smallmatrix} f(r) & g(r) \\ g(r) & h(r) \end{smallmatrix}\right)$ for functions f, g, $h : F \to F$. Then

$$\begin{aligned}\overline{A(r)} &= \{(a, b, af(r) + bg(r), ag(r) + bh(r) \mid a, b \in F\} \\ &= \langle(1, 0, f(r), g(r)), (0, 1, g(r), h(r))\rangle\end{aligned}$$

corresponds to the point $\langle(1, g(r), h(r), f(r), -g(r), g(r)^2 - h(r)f(r))\rangle$. Each such point lies in the hyperplane $x_2 + x_5 = 0$, and this is just the five-dimensional space V mentioned above. Also, $\overline{A(\infty)}$ corresponds to the point $\langle(0, 0, 0, 0, 0, 1)\rangle$.

Any BLT set in V is orthogonally equivalent to one containing $\langle(0, 0, 0, 0, 0, 1)\rangle$, and then will consist of q further points $\langle(1, g(r), h(r), f(r), -g(r), g(f)^2 - h(r)f(r))\rangle$ for suitable functions f, g, h. The definition of a BLT set S is that $\langle u, v, w\rangle^\perp$ is anisotropic for any distinct $u, v, w \in \mathrm{S}$; and this is equivalent to the condition that $-\det(M_r - M_s)$ is a nonsquare for all distinct $r, s \in F$ ([BLT]; the proof is implicit in that of the above lemma).

Under the Klein correspondence S produces a family $\bar{\mathscr{F}}$ of $q+1$ totally singular lines $\overline{A(r)} = \{(u, uM_r) \mid u \in F^2\}$ and $\overline{A(\infty)}$. These, in turn, determine a family $\mathscr{F}$, simply by taking $B_r = \frac{1}{2}M_r$. If $\{(u, uC_ru^t, uM_r) \mid u \in F^2\}$ is a member of a second family $\mathscr{F}'$ projecting onto $\bar{\mathscr{F}}$, then $C_r + C_r^t = M_r$ and hence (as noted earlier) $uB_ru^t = uC_ru^t$; thus, $\mathscr{F}' = \mathscr{F}$. ∎

Various aspects of the above theorem are implicit in [BLT] and [PR], but viewed from a rather different point of view. For example, [PR, Section IV] contains the result (iii) restricted to suitable stabilizers within $P\Gamma Sp(\bar{Q})_{\mathscr{F}}$ and $P\Gamma O(V)_{S(\mathscr{F})}$—and phrased in a somewhat different manner. The point of (iii) is that the automorphism group of the generalized quadrangle $\mathbf{Q}(\mathscr{F})$—having $\mathscr{F}$ as a distinguished point—generally can be computed entirely within the subgroup $P\Gamma O(V)_{S(\mathscr{F})}$ of the projective orthogonal group $P\Gamma O(V)$. For, provided that the quadrangle is non-classical, its automorphism group $\mathrm{Aut}\,\mathbf{Q}(\mathscr{F})$ is $Q \rtimes \Gamma Sp(\bar{Q})_{\mathscr{F}}$; since $\mathrm{Aut}\,\mathbf{Q}(\mathscr{F})$ contains the group Z^* inducing scalar transformations of $\bar{Q}$ it follows that only $Q \rtimes \Gamma Sp(\bar{Q})_{\mathscr{F}} = Q \rtimes \Gamma Sp(\bar{Q})_{\mathscr{F}}$—and hence only $P\Gamma O(V)_{S(\mathscr{F})}$—must be determined. (N.B. The preceding assertion concerning $\mathrm{Aut}\,\mathbf{Q}(\mathscr{F})$ is almost, but not quite, contained in [Ka3, Lemma 3.1]. It is not difficult to push that lemma slightly further in order to obtain the assertion.)

Translation planes. The translation planes associated with $\mathbf{S}$ depend on a choice of a member u of $\mathbf{S}(\mathscr{F})$ (or, equivalently, a member of $\mathscr{F}$). Namely, the translation plane $\mathbf{T}(\mathbf{S}(\mathscr{F}), u)$ associated in [FT] with the pair $\mathbf{S}(\mathscr{F})$, u arises from the spread of lines of $\bar{Q}$ corresponding, under the Klein correspondence, to the set of q^2+1 singular points in

$$\bigcup \{(u^\perp \cap V \cap v)^\perp \mid v \in \mathbf{S}(\mathscr{F}) - \{u\}\} = \bigcup \{\langle u, v, V^\perp\rangle \mid v \in \mathbf{S}(\mathscr{F}) - \{u\}\}.$$

"Embedding." Since the passage from $\mathscr{F}$ to the planes $\mathbf{T}(\mathbf{S}(\mathscr{F}), u)$ is natural, it is easy to make it a bit more explicit. Starting with $\mathscr{F}$ and the choice of some $A \in \mathscr{F}$ (corresponding to u), $\mathbf{T}(\mathbf{S}(\mathscr{F}), u)$ can be obtained by means of the following steps.

(1) For each $B \in \mathscr{F} - \{A\}$, let $R(A, B)$ consist of the set of totally isotropic lines of $\bar{Q}$ meeting both $\bar{A}$ and $\bar{B}$; this is a regulus.

(2) Form the opposite regulus $R^{\#}(A, B)$ of $R(A, B)$ (no longer consisting of totally isotropic lines of $\bar{Q}$); clearly, $\bar{A}, \bar{B} \in R^{\#}(A, B)$.

(3) The desired spread is $\Sigma := \bigcup \{R^{\#}(A, B) \mid B \in \mathscr{F} - \{A\}\}$.

(4) Each member of Σ is a 2-space $T/Z(Q)$ of $\bar{Q}$ for a uniquely determined subgroup T of Q having order q^3. The points of $\mathbf{T}(\mathbf{S}(\mathscr{F}), u)$ can be identified with the orbits of $Z(Q)$ on the set of q^5 points of $\mathbf{Q}(\mathscr{F})$ not collinear with the point $\mathscr{F}$, while the lines of $\mathbf{T}(\mathbf{S}(\mathscr{F}), u)$ can be identified with the point-orbits of the q^2+1 subgroups T.

It should be noted that the only ones of these groups T that are abelian are those for which $T/Z(Q) \in \mathscr{F}$.

Remark. It is difficult to get too enthusiastic about the above "embedding." *Every* translation plane arising from a spread of the 4-space $\bar{Q}$ gives rise to an "embedding" into the *same* generalized quadrangle $\mathbf{Q}(\mathscr{F})$ by the method described in (4). The only difference in (4) is that the spread is somehow related to the family $\mathscr{F}$. Understanding this difference seems to require a description of the reguli $R(A, B)$ and $R^{\#}(A, B)$, and the orbits of the groups T, in terms of configurations within the generalized quadrangle.

Characteristic 2. The previous methods break down when q is even. In that case, the $q+1$ groups $\overline{A(r)}$ *form* a regulus. Since the squaring map $gZ(Q) \to g^2$ from $\bar{Q}$ to $Z(Q)$ is a quadratic form, this regulus is one of only two available within $\bar{Q}$. Thus, the entire group $A(r)$ needs to enter into the discussion when q is even (previously we were able to deal only with $\overline{A(r)}$). Nevertheless, it seems as if there should be a suitable type of modification of the present approach in that case as well.

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