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Finite Groups with a Split BN-Pair of Rank 1. II*

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In Part I of this paper (Hering, Kantor, and Seitz [50]), 2-transitive groups of even degree were classified when the stabilizer of a point has a normal subgroup regular on the remaining points. The identification with groups of known type was made by finding a 2-Sylow subgroup and then applying the deep classification theorems of Alperin, Brauer and Gorenstein [1, 2] and Walter [39].

The purpose of the present continuation of [50] is to point out that the proof of the main result of [50] can be completed without using [1] and [2]. Moreover, Walter's classification theorem [39] and the Gorenstein–Walter Theorem [49] are not required in [50], although the end of Walter [53] seems to be needed.

Our arguments are natural continuations of those of [50, Sections 4, 8, and 9]. Much use is also made of character-theoretic information contained in Brauer [46] and [47]. Our goal is to show that a minimal counterexample has a cyclic two points stabilizer $G_{\alpha\beta}$ and then apply a result of Kantor, O'Nan and Seitz [22, Theorem 1.1 or Section 5, Case D]. We first show that $G_{\alpha\beta}$ is metacyclic, and then "transfer out field automorphisms" in order to prove that $G_{\alpha\beta}$ is cyclic.

This transfer argument yielded an unexpected dividend: in the course of examining a similar argument in Suzuki [34, Section 21], an error was found. This has been corrected, and, in fact, the entire transfer argument is stated for odd and even degree groups simultaneously.

The numbering of both the sections and the references will be continued from [50].

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11. Preliminary Lemmas

Lemma 11.1. *Let p be an odd prime and x a positive integer such that $x \equiv -1 \pmod{p}$. Then $(x^p + 1)_p/(x+1)_p = p$.*

Proof. Set $x = -1 + yp$. Then

$$x^p + 1 = (-1+yp)^p + 1 = -1 + p \cdot yp - \binom{p}{2} y^2p^2 + \cdots - y^pp^p + 1$$

$$= yp\left(p - \binom{p}{2} yp + \cdots - y^{p-1}p^{p-1}\right).$$

Thus, $(x^p + 1)/(x+1) \equiv p \pmod{p^2}$.

Lemma 11.2. *Let P be a p-Sylow subgroup of a group G and $N = N_G(P)$. Suppose that P contains no section isomorphic to $Z_p \wr Z_p$. Then $G/O^p(G) \approx N/O^p(N)$.*

Proof. This version of the Hall–Wielandt Theorem follows from the Proof of Theorem 14.4.1 in [14].

Lemma 11.3. *Let G be a finite group having no normal subgroup of index 2. Suppose that a 2-Sylow subgroup S of G is quasidihedral. Then the principal 2-block $B_0(2, G)$ of G consists of characters $\chi_0 = 1_G, \chi_1, \chi_2, \chi_3, \chi_4,$ and characters $\chi^{(j)}$ of the same degree x such that, if $x_i = \chi_i(1)$, there are signs $\delta_1, \delta_2, \delta_3$ and an integer $m \equiv 1 \pmod 4$ such that the following hold:*

(i) $\chi_1(t) = \delta_1 m$, $\chi_2(t) = -\delta_2 m$ *and* $\chi_3(t) = -\delta_3$ *for any involution* t;

(ii) $1 + \delta_1 x_1 = \delta_1 x = -\delta_2 x_2 - \delta_3 x_3$, $1 + \delta_2 x_2 = \delta_2 x_4$;

(iii) $x_1 \equiv \delta_1(2 - m)$, $x_2 \equiv -m$, $x_3 \equiv -\delta_3 + \frac{1}{2}|S| \pmod{|S|}$ *and* $m \equiv 1 + \frac{1}{4}|S| \pmod{\frac{1}{2}|S|}$;

(iv) $x_4 \equiv x \equiv 0 \pmod 2$;

(v) $\delta_1\delta_2\delta_3 = 1$, $x_1 x_2 = m^2 x_3$;

(vi) $\chi_i(s) = \delta_i$, *where* $|S : \langle s\rangle| = 2$ *and* $i = 1, 2, 3$;

(vii) $|G|$ *divides* $|C(t)|^3 x_1(x_1 + \delta_1)(m+1)$; *and*

(viii) *If* $\{k, \ell\} = \{1, 2\}$ *and* $\delta_k = 1$ *then* $h = m^2 - x_\ell > 0$, $\delta_\ell = -1$, $x_k = m^2(m^2 - h - 1)/h$, *and* $x_3 = (m^2 - h)(m^2 - h - 1)/h$.

Proof. (i)–(vii) are found in Brauer [47, Section VIII]. It is straightforward to deduce (viii) from (i)–(v) [1, Proposition 3.2.8].

LEMMA 11.4. *Let G be a finite group having no normal subgroup of index 2 whose 2-Sylow subgroups are wreathed $Z_{2^e} \wr Z_2$. Let $\langle t, v\rangle$ be any Klein group in G.*

(i) *$B_0(2, G)$ consists of characters of degrees 1, ϵm^3, $m(m+1)$, m^2+m+1, $\epsilon m(m^2+m+1)$, $\epsilon(m+1)(m^2+m+1)$, and $\epsilon(m-1)(m^2+m+1)$, where m is an odd integer, $\epsilon = \pm 1$ and $\epsilon m > 0$.*

(ii) *There are integers a, c, h such that*

$$|G| = a^3 ch\epsilon m^3(m+1)(m-1)(m^2+m+1),$$
$$|C(t)| = ach\epsilon(m+1)m(m-1),$$
$$|C(\langle t, v\rangle)| = ch\epsilon(m-1).$$

Proof. This is all stated in Brauer [46] except for the fact that m is odd, which is clear since $|G : C(t)|$ is odd.

12. REVIEW

We now continue the proof of Theorem 1.1. We will not assume that Theorem 7.8 holds. The following situations were arrived at in [50].

A minimal counterexample to Theorem 1.1 is simple (Theorem 7.7). A 2-Sylow subgroup S of G is either (I) quasidihedral or wreathed, or (II) elementary abelian of order 8 (see the first part of the proof of Theorem 7.8). It follows that all involutions are conjugate to an involution $t \in G_{\alpha\beta}$. Here $C_0(t)^\Delta = PSL(2, q)$ for some power q of an odd prime q_0 (Sections 5–7). We may assume that $t \in Z(S)$ and $S_{\alpha\beta}$ is a 2-Sylow subgroup of $G_{\alpha\beta}$.

The only places where Theorem 7.8 was used were in Lemmas 8.4, 9.2(ii), and 9.6. Consequently, we have the following cases.

Case I. Here $C_0(t) = SL(2, q)$.

Case II. Here $C_0(t) = PSL(2, q)$ and $|C(t)^\Delta : C_0(t)^\Delta|$ is odd. Also, $q > 3$ by Theorem 5.1, $q \equiv 3 \pmod 4$, and $|S_{\alpha\beta}| = 2$ (see Cases 2 and 4 at the end of the proof of Theorem 8.9).

13. CASE I BEGUN

In Sections 13–15 we will consider Case I of Section 12. In this section $n = |\Omega|$ and the structure of $G_{\alpha\beta}$ will be determined.

Let θ be the permutation character of degree $n-1$. Then $\theta \in B_0(2, G)$

[22, Lemma 3.6]. Clearly $\theta(1)$ is odd and $\theta(t) > 1$. We will use the notation of Lemmas 11.3 and 11.4.

In the quasidihedral case, $q = \pm m$ and $\theta = \chi_1$ or χ_2. Once we have shown in Lemma 13.1 that $n - 1 \equiv 1 (\text{mod } 4)$ it will follow that $\delta_1 = 1$, $m = q$ and $\theta = \chi_1$.

In the wreathed case, $n - 1 = \epsilon m^3, m^2 + m + 1$ or $\epsilon m(m^2 + m + 1)$.

LEMMA 13.1. (i) *$C(t)^{\Delta}$ contains $PGL(2, q)$ as a subgroup of odd index f.*

(ii) *$S_{\alpha\beta}$ is a 2-Sylow subgroup of $G_{\alpha\beta}$, and is cyclic of order* >4.

(iii) *If S is quasidihedral then $n - 1 \equiv q \equiv 1(\text{mod } 4)$, while if S is wreathed then $n - 1 \equiv q \equiv 3(\text{mod } 4)$.*

(iv) $q > 3$.

Proof. Recall that $n - 1 \equiv q (\text{mod } 4)$ (Lemma 4.1). Let E and F be as in Lemma 9.2. Then E is generalized quaternion and F is cyclic (Lemma 9.2(ii)). We distinguish the two possibilities (a) $|F| = 2$ and (b) $|F| \geqslant 4$.

(a) There is an involution $v \in S - E$. By Lemma 9.3(iii), $C(t)^{\Delta}$ contains $PGL(2, q)$. Thus, S^{Δ} is a semidirect product of a dihedral group of order $(q^2 - 1)_2$ and a cyclic group. Since $S^{\Delta} \approx S/\langle t \rangle$ it follows that S is quasidihedral and (i) holds.

Suppose that $q \equiv 1(\text{mod } 4)$. Then v^{Δ} is regular, so that $\Omega_1(S_{\alpha\beta}) = \langle t \rangle$. Also, $S_{\alpha\beta}$ is cyclic of order $\geqslant 2|S^{\Delta}_{\alpha\beta}| > 4$, and (ii) holds.

Now suppose that $q \equiv 3(\text{mod } 4)$. Then $S_{\alpha\beta}$ is a Klein group. By Lemma 4.5, $\underset{\sim}{Q}$ is abelian of order q^3. In the notation of Lemma 11.3, $q^3 = x_1$ or x_2. If $\theta = \chi_2$, then by Lemma 11.3(iii) we have $q^3 = x_2 \equiv \pm q (\text{mod } |S|)$. Since $|S| \geqslant 2|C_0(t)|_2$ this is impossible.

Thus, $-1 \equiv q^3 \equiv x_1 \equiv \delta_1(\text{mod } 4)$, so that

$$\delta_1 = -1 \qquad \text{and} \qquad |G| \mid |C(t)|^3 q^3 (q^3 - 1)(q - 1).$$

Since $9 \nmid q^2 - q + 1$ there is a prime $\ell \mid q^2 - q + 1, \ell \neq 3$. Then $q^3 + 1 \mid |G|$ implies that $\ell \mid |C(t)|$. An element $x \in C(t)$ of order ℓ fixes at least three points of Δ. Also, $\ell \nmid q^3 - q$, so that x fixes a point of $\Omega - \Delta$. Since $\underset{\sim}{Q}$ is abelian and $C_0(t) \cap C(x) = SL(2, q')$ for some q', this is impossible by Lemma 4.4.

(b) Here S is wreathed since EF is a central product. We first show that (i) holds. As $C_0(t)^{\Delta} = PSL(2, q)$, $Z(S/F)$ has order 2 and $Z(S) \leqslant EF$. Thus, $Z(S) \leqslant Z(EF) = F$. As $S/Z(S)$ is dihedral, it follows that $Z(S) = F$ and S/F is dihedral. By Lemma 9.4, $S > EF$. Then $(C_0(t)S)^{\Delta} \geqslant PGL(2, q)$, $P\hat{G}L(2, q)$, or $PSL(2, q)\langle a^{\Delta} \rangle$ where a^{Δ} is an involutory field automorphism. However, S^{Δ} is dihedral, so that (i) holds.

Clearly, $S_{\alpha\beta}^{\Delta}$ is cyclic and hence $S_{\alpha\beta}$ is abelian. Suppose that $S_{\alpha\beta}$ is cyclic. Since $EF - \langle t \rangle$ contains an involution v, we must have $q \equiv 3 (\mathrm{mod}\ 4)$. Thus, (i)–(iii) hold in this situation.

Now suppose that $S_{\alpha\beta}$ is noncyclic. By Lemma 4.5, Q is abelian of order q^3. We have seen that $F = Z(S)$ and S/F is dihedral. Now from the structure of S it follows that each involution v in S is in EF, and since F is cyclic, $C(t)$ contains a single class of involutions other than t. As $v \in S_{\alpha\beta}$, $q \equiv 1 (\mathrm{mod}\ 4)$.

In the notation of Lemma 11.4, $q^3 = \epsilon m^3$ or $m^2 + m + 1$. If $q^3 - 1 = m^2 + m$ then $q^3 - 1 \mid |C(t)|$. Letting ℓ be a prime such that $3 \neq \ell \mid q^2 + q + 1$, as in (a) we again obtain a contradiction. Thus, $q = \epsilon m$. Suppose that $q = m$. Then $|G|$ divides $|C(t)|^3 q^3 (q + 1)(q^3 - 1)$, which again leads to a contradiction. Consequently, $q = -m$ and $|C(\langle t, v \rangle)| = ch(q + 1)$.

On the other hand, $v^{\Delta} \in C_0(t)_{\alpha\beta}^{\Delta}$, so that $|C(t) \cap C(v)|$ divides $2(q - 1)|C_W(v)| f$. Thus, $\frac{1}{2}(q + 1) \mid |C_W(v)| f$. Here $C_W(v)$ is faithful and semiregular on $\Delta(v) - \{\alpha, \beta\}$, so that $|C_W(v)| \mid (q - 1)$. Since $\frac{1}{2}(q + 1)$ is odd we must have $\frac{1}{2}(q + 1) \mid f$. However, $q = q'^f$ for some q'. This contradiction completes the proof of (i)–(iii).

Finally, if $q = 3$, then $|S| = 32$, $C(t)$ is solvable by the Feit–Thompson Theorem, and, hence, $G \approx PSU(3, 3)$ (Fong [48]), which is not the case.

LEMMA 13.2. (i) $t \in Z(G_{\alpha\beta})$.

(ii) $|G : C(t)| = n(n - 1)/(q + 1)q$ *and* $|G_\alpha : C(t)_\alpha| = (n - 1)/q$.

Proof. If $t \notin Z(G_{\alpha\beta})$ then $x^t = x^{-1}$ for some $x \in G_{\alpha\beta}^{\#}$ of odd prime order r. Suppose that $C_Q(x) = 1$. Applying the Brauer–Wielandt Theorem [41] to the dihedral group $\langle t, x \rangle$ acting on Q, we find that $|C_Q(t)|^{2r} = |Q|^r$, contradicting Theorem 6.1.

Thus, $|\Delta(x)| > 2$. Since $t \in N(\langle x \rangle) - C(x)$, $|C(x)_{\alpha\beta}|$ is odd. In view of the 2-Sylow subgroups of G, by Lemma 4.4 we must have $C_0(x) = PSL(2, \ell)$ with $\ell \equiv 3 (\mathrm{mod}\ 4)$. If u is an involution in $C_0(x)$ then $\Delta(u) \cap \Delta(x) = \phi$, so that $r = |x| \mid |\Delta(u)| = q + 1$.

Let R be an r-Sylow subgroup of $G_{\alpha\beta}$ normalized by $S_{\alpha\beta}$. Suppose that $|\Delta(R)| > 2$. x is conjugate to an element of R, so that as above, $C_0(R) = PSL(2, \ell')$ with $\ell' \equiv 3 (\mathrm{mod}\ 4)$. Since $S_{\alpha\beta}$ is cyclic of order ≥ 4 we must have $t \in W_R$. Now $C_0(R) \times W_R$ contains an elementary abelian subgroup of order 8, which is not the case.

Thus, $C_Q(R) = 1$, so that $r \mid |Q^{\#}| = n - 2$. It follows that

$$0 \equiv n - (\ell + 1) \equiv 2 - (\ell + 1) \ (\mathrm{mod}\ r).$$

Since $C_0(x)\langle t \rangle = PGL(2, \ell)$, there is an element $y \in C_0(x)_{\alpha\beta} \cap C(t)$ of order r. Here y acts on $\Delta - \{\alpha, \beta\}$ and $\Omega - \Delta$, where $|\Delta - \{\alpha, \beta\}| = q - 1 \equiv -2 (\mathrm{mod}\ r)$ and $|\Omega - \Delta| = n - (q + 1) \equiv 2 (\mathrm{mod}\ r)$. From Lemma 4.4 it follows that $C_0(y)^{\Delta(y)} = PSU(3, q')$, where $q = q'^i$ for some integer i. Then $0 \equiv n - (q'^3 + 1) \equiv 2 - (q'^3 + 1) (\mathrm{mod}\, r)$, so that $q^3 = q'^{3i} \equiv 1 \ (\mathrm{mod}\, r)$. However, $q \equiv -1 \ (\mathrm{mod}\, r)$, a contradiction.

This proves (i), and (ii) follows readily.

LEMMA 13.3. (i) *$C_0(t)_{\alpha\beta} S_{\alpha\beta}$ is cyclic of order $\geqslant 2(q - 1)$.*

(ii) *$O(C_0(t)_{\alpha\beta})$ is fixed-point-free on Q.*

(iii) *If $q - 1$ is not a power of 2 then Q is nilpotent.*

Proof. (i) $C_0(t)_{\alpha\beta}$, $S_{\alpha\beta}$, and $(C_0(t)S_{\alpha\beta})^{\Delta}$ are all cyclic.

(ii) If $1 \neq x \in C_0(t)_{\alpha\beta}$ then x is inverted in $C_0(t)$ and centralizes $S_{\alpha\beta}$. By Lemma 4.3, $|\Delta(x)| = 2$.

(iii) This follows from (ii) and a theorem of Thompson [37].

LEMMA 13.4. (i) *Each prime divisor of f divides n.*

(ii) *$(f, q(q - 1)) = 1$.*

Proof. (i) Let p be a prime dividing f but not n. Suppose that $G_\alpha > X \geqslant Y = C_0(t)_{\alpha\beta} S_{\alpha\beta} WQ$, where G_α / X is a p-group. $G_\alpha = X\langle a \rangle$, where we may assume that $|\Delta(a)| \geqslant 3$. If $a^g \in G_\alpha$, $g \in G$, we claim that $a \equiv a^g (\mathrm{mod}\, X)$. Let $a^{gh} \in G_{\alpha\beta}$ for $h \in Q$. Then $a^{gh} = a^d$, $d \in G_{\alpha\beta}$ (Lemma 4.3). Here $a^{-1}a^d \in X$, so that $a^g \equiv a^{gh} = a^d \equiv a (\mathrm{mod}\, X)$.

Since $(|G : G_\alpha|, |G_\alpha / X|) = 1$, it follows that the image of a under the transfer map $G \to G_\alpha / X$ is nontrivial, contradicting the simplicity of G.

(ii) Clearly $q \mid n - 1$. By Lemma 13.3 (ii), $(q - 1)_{2'} \mid |Q^{\#}|$. Thus, (ii) follows from (i).

LEMMA 13.5. *Let $t' = (\alpha\beta) \cdots$ be an involution, and suppose that t' inverts b elements of $O(W)$. Then $n - 1 = q(b(q^2 - 1) + 1)$.*

Proof. If $u \neq t$ is an involution in $C(t)$ then u^{Δ} is a regular involution. There are $(q - 1)/2$ such involutions in $C(t)^{\Delta}$ interchanging α and β, all of which are conjugate in $C(t)^{\Delta}$. Suppose that $u^{\Delta} = t'^{\Delta}$. Then $ut' \in W$, and t' inverts ut'. Thus, there are $\frac{1}{2}(q - 1)b^*$ involutions $(\alpha, \beta) \cdots$, where b^* is the number of elements of W inverted by t'. However, $C_W(t')$ contains a 2-Sylow subgroup of W. Thus, $b^* = 2b$.

There are $(n - 1)(q - 1)b$ involutions moving α. Since this number is also $n(n - 1)/(q + 1)q - (n - 1)/q$, the lemma follows.

LEMMA 13.6. *Let $A \neq 1$ be a normal subgroup of G_α contained in Q. If $(q, |A|) = 1$ then $G_{\alpha\beta}$ is fixed-point-free on A.*

Proof. Let $x \in G_{\alpha\beta}^{\#}$ have prime order r and suppose that $C_{\Delta}(x) \neq 1$. By Lemma 13.3, $r > 2$ and $x \notin C_0(t)_{\alpha\beta}$. Assume first that $|\Delta \cap \Delta(x)| \geqslant 3$. Then $C_0(t) \cap C(x)$ is $SL(2, q')$ with $q' \mid q$. Since $C_{\Delta}(x) \neq 1$ this is impossible by Lemma 4.4.

Thus, $|\Delta \cap \Delta(x)| = 2$. Since $G_{\alpha\beta} = C_0(t)_{\alpha\beta} S_{\alpha\beta} G_{\alpha\beta\delta}$, $\delta \in \Delta - \{\alpha, \beta\}$, we have $x = cy$ with $1 \neq c \in C_0(t)_{\alpha\beta}$, $1 \neq y \in G_{\alpha\beta\delta}$ and $|c| = |y| = r$. Then $r \mid q - 1$, so $y \in W$ by Lemma 13.4.

Now consider $C_0(x)$. By Lemmas 4.4 and 13.2, $C_0(x)$ must be $PSL(2, l)$ for some l. There is an involution $v \in C_0(x)$ interchanging α and β. Then v normalizes $C_0(t)_{\alpha\beta}$ by Lemma 13.2. Consequently, v inverts c and we have $cy = (cy)^v = c^{-1}y^v$ where $y^v \in W$. However, $c \neq 1$ has odd order, so this is impossible.

LEMMA 13.7. $n = q^3 + 1$.

Proof. By Lemma 13.3 there is a q_0-Sylow subgroup $Q_0 \geqslant C_Q(t)$ of Q normalized by $C_0(t)_{\alpha\beta} S_{\alpha\beta}$. Since $C_0(t)_{\alpha\beta} S_{\alpha\beta}$ is fixed-point-free on $N_{Q_0}(C_Q(t))/C_Q(t)$ and has order $\geqslant 2(q - 1)$, we find that $|Q_0| = q$ or $|Q_0| \geqslant q^3$.

First suppose that S is quasidihedral. We have $\theta = \chi_1$, $q = m$ and $\delta_1 = 1$ by the remark preceding Lemma 13.1. Then $h = q^2 - x_2 > 0$ and $x_1 = q^2(q^2 - h - 1)/h$. If $|Q_0| = q$ we can write $h = qh'$ and $x_1 = q(q^2 - qh' - 1)/h' < q^3$, whereas $x_1 \geqslant q^3$ by Lemma 13.5. Since $x_1 < q^4$ we must have $|Q_0| = q^3$. Then $q \mid q^2 - h - 1$, so that $h \geqslant q - 1$ and $x_1 \leqslant q^3$. Thus, $x_1 = q^3$.

Suppose now that S is wreathed. By Lemmas 13.3 and 13.1 (iii), (iv), $Q = Q_0 \times A$ with A an abelian q_0'-group. Let $A \neq 1$. By Lemma 13.6, $G_{\alpha\beta}$ is fixed-point-free on A. If $|Q_0| \geqslant q^3$ then by Lemma 13.5 we have

$$b < |W| < |G_{\alpha\beta}| < |A| = |Q|/|Q_0| \leqslant (b(q^2 - 1) + 1)/q^2 < b.$$

Thus, Q_0 must be $C_Q(t)$, Q is abelian, and the argument in [4, Satz 3.15] or [22, Lemma D.5] shows that G is not simple.

Consequently, $Q = Q_0$ and $n - 1$ is a power of a prime. By Lemma 11.4, $n - 1 = \epsilon m^3$ or $m^2 + m + 1$. Also, by Lemma 13.2 $n(n - 1)/(q + 1)q = |G : C(t)| = a^2m^2(m^2 + m + 1)$. Thus, $n - 1 = \epsilon m^3$. If $q = |m|$ the lemma is immediate.

Assume that $q < |m|$. Then $|C(t)|_{q_0} \geqslant |m| > q$. We may assume that $v = t'$ in Lemma 11.4. Let L be a q_0-Sylow subgroup of $G_{\alpha\beta}$ normalized by v. By Lemma 13.4, $L \leqslant W$. Since $qq_0 \mid |Q| = q(b(q^2 - 1) + 1)$, $q_0 \nmid b$. Thus, $L \leqslant C(\langle t, v\rangle)$, so that

$$|C(\langle t, v\rangle)|_{q_0} = |C(\langle t, v\rangle)_{\alpha\beta}|_{q_0} = |W|_{q_0} = q^{-1}|C(t)|_{q_0}.$$

By Lemma 11.4, $q = |C(t) : C(\langle t, v\rangle)|_{q_0} = (a(m+1)m)_{q_0} > q$, which is absurd. This proves the lemma.

We now list the properties of G to be used in Section 15.

THEOREM 13.8. *Set $q = q_0^e$ and $\Delta = \Delta(W)$.*

(i) *$G = O^{2'}(G)$ and G does not satisfy the conclusions of Theorem* 1.1.

(ii) *If $X \subset G$ fixes $\geqslant 3$ points then $C(X)^{\Delta(X)}$ satisfies the conclusions of Theorem* 1.1.

(iii) *$n = q^3 + 1$ and $|\Delta| = q + 1$.*

(iv) *$N(W)^{\Delta}$ is a subgroup of $P\Gamma L(2, q)$ containing $PGL(2, q)$ as a subgroup of odd index $f \mid e$.*

(v) *W is a nontrivial weakly closed subgroup of $G_{\alpha\beta}$.*

(vi) *W is the pointwise stabilizer of Δ.*

(vii) *W is semiregular on $\Omega - \Delta$.*

(viii) *W centralizes each involution $(\alpha\beta)\cdots$.*

(ix) *$|W| \mid q + 1$.*

(x) *$G_{\alpha\beta}$ has a weakly closed subgroup $D > W$ such that $|D : W| = q - 1$, $G_{\alpha\beta}/D$ is cyclic of order f, and D^{Δ} is contained in $PGL(2, q)$.*

(xi) *$C_Q(W) \lhd Q$ and D is fixed-point-free on $Q/C_Q(W)$.*

(xii) *D is cyclic.*

(xiii) *No element of $W - \langle t\rangle$ is inverted in G.*

(xiv) *$G_{\alpha\beta} > D$.*

Proof. (i)–(iv) are already known. (v) follows from Lemmas 13.2 and 4.3. (vi) is clear.

(vii) If $w \in W^{\#}$ and $\Delta(w) \supset \Delta$ then, by Lemma 4.4, $|\Delta(w)| = q^3 + 1 = n$, which is absurd.

(viii) This follows from Lemma 13.5 as $F \leqslant (Z)S$.

(ix) By (vii) and (viii), if $t' = (\alpha\beta)\cdots$ is an involution then W is semiregular on $\Delta(t')$, so that $|W| \mid q + 1$.

(x) $D = C_0(t)_{\alpha\beta}S_{\alpha\beta}W$ meets all the requirements.

(xi) Since $(|C_0(t)_{\alpha\beta}S_{\alpha\beta}|, |W|) = 2$ or 4, D is fixed-point-free on $N_Q(C_Q(t))/C_Q(t)$. Here $2(q-1) \mid |D|$ and $|Q : C_Q(W)| = q^2$. Thus, $C_Q(W) \lhd Q$.

(xii) We have just seen that D acts irreducibly on $Q/C_Q(W)$. Since $C_0(W)_{\alpha\beta} \leqslant Z(D)$ and $|C_0(W)_{\alpha\beta}| = q - 1$, this representation can be viewed as a 2-dimensional $GF(q)$-representation. Since each Sylow subgroup of D is cyclic, it suffices to show that D is abelian. By Lemma 13.3(ii), we may assume that $|W| > 2$.

Suppose that D is nonabelian. Then, D is an absolutely irreducible subgroup of $GL(2, q)$, so that $C_0(W)_{\alpha\beta} = Z(GL(2, q))$. Then $D/C_0(W)_{\alpha\beta}$ is isomorphic to a subgroup of $PGL(2, q)$, has order dividing $q + 1$, and has cyclic Sylow groups. Thus $D/C_0(W)_{\alpha\beta}$ is cyclic or is dihedral of order twice an odd number.

If $D/C_0(W)_{\alpha\beta}$ is cyclic, then so is D and (xii) holds. Suppose that $D/C_0(W)_{\alpha\beta}$ is dihedral. Then $C_0(W)_{\alpha\beta}W/C_0(W)_{\alpha\beta}$ is its cyclic subgroup of index 2, so that $C_0(W)_{\alpha\beta}W$ is cyclic. If $q \equiv 1 (\text{mod } 4)$ each element of $C_0(W)S_{\alpha\beta}W - C_0(W)W$ must invert W. By (viii) we must have $q \equiv 3 (\text{mod } 4)$. Then $|W \cap S_{\alpha\beta}| \geqslant 4$ implies that $|D/C_0(W)_{\alpha\beta}|_2 \geqslant 4$, which is not the case.

(xiii) By (xii) no element of $W - \langle t \rangle$ is inverted in $G_{\alpha\beta}$, and (xiii) follows from Lemma 4.3.

(xiv) If $G_{\alpha\beta} = D$ then $G_{\alpha\beta}$ is cyclic. By [22, Theorem 1.1 or Section 5, Case D], it follows that G is $PSU(3, q)$, which we have assumed is not the case.

14. Remarks on Suzuki's Paper [34]

We digress from the even degree case of Theorem 1.1 in order to discuss the important part of the odd degree case due to Suzuki [34]. There is an error in [34, p. 577, lines 3–4], as can be seen from our Lemma 11.1 or by considering $P\Gamma U(3, q)$. This error is due to [34, Lemma 38(iii)].

In Section 15 we will consider both the even and odd degree cases of Theorem 1.1. As a result we will prove [34, Lemma 60].

First, it is necessary to note that (i)–(xiv) of Theorem 13.8 again hold. (i) and (xiv) are assumed in the proof of [34, Lemma 60]. (ii) is found in [34, Section 8] (iii) is Lemma 59, while (iv)–(xiii) follow from [34, Lemma 31, Theorem 5, Section 14, and Lemma 49].

15. Case I Completed

The following result will complete Case I and correct the error in Suzuki [34] mentioned in Section 14.

Theorem 15.1. *If G is a finite group 2-transitive on a set Ω such that, for $\alpha \in \Omega$, G_α has a normal subgroup Q regular on $\Omega - \alpha$, then there are no $\Delta \subset \Omega$ and $W < G$ such that conditions* (i)–(xiv) *of Theorem* 13.8 *hold.*

Proof. We shall "transfer out" part of $G_{\alpha\beta}$, thereby contradicting the fact that $O^{2'}(G) = G$. Clearly,

$$(*) \qquad |G| = (q^3 + 1)q^3(q - 1)|W|f.$$

Recall that Q is a q_0-group. Let p be a prime divisor of f. Precisely as in Lemma 13.4, $p \mid q^3 + 1$, and in particular $q_0 \nmid f$. Let $q = \ell^p$. If $X \leqslant G_{\alpha\beta}$ we can define $C_0(X)$ as in Lemma 4.2. Then Lemma 4.3 still holds.

We will frequently use the fact that a central extension of $PSU(3, q)$ by a group of order a prime $\neq 2, 3$ splits (see the proof of [34, Lemma 58]).

LEMMA 15.2. (i) *If X is a subgroup of $G_{\alpha\beta}$ fixing $\geqslant 3$ points of Δ and such that $X \cap W = 1$, then $N(X) = C(X)$.*

(ii) *Either $p \mid q^2 - q + 1$ or $p \mid |W|$ and a p-Sylow subgroup of $G_{\alpha\beta}$ is noncyclic.*

(iii) *If either $p \neq 3$ or $p = 3$ and a p-Sylow subgroup $P_{\alpha\beta}$ of $G_{\alpha\beta}$ is noncyclic, then $P_{\alpha\beta}$ has a subgroup P_0 of order p such that $|\Delta(P_0)| = \ell^3 + 1$ and $C_0(P_0)^{\Delta(P_0)} = PSU(3, \ell)$.*

Proof. (i) This follows from Lemma 4.3 and $N(X)^{\Delta}_{\alpha\beta} = C(X)^{\Delta}_{\alpha\beta}$.

(ii) If $p \nmid q^2 - q + 1$ then $3 \neq p \mid q + 1$ and, by $(*)$, $C_0(W)G_{\alpha\beta}$ contains a p-Sylow subgroup P of G. If $G_{\alpha\beta}$ has a cyclic p-Sylow subgroup then P is metacyclic and nonabelian (Lemma 11.1). A result of Huppert [51] then contradicts the fact that $G = O^{2'}(G)$.

(iii) A p-Sylow subgroup X of $G_{\alpha\beta}$ acts on $Q/C_Q(W)$. If $p \mid q^2 - q + 1$ but $p \neq 3$, then $p \nmid q^2 - 1$, and hence $C_Q(X) \nleqslant C_Q(W)$. If $p \mid q + 1$ and X is noncyclic then X has a subgroup P_0 of order p with $C_Q(P_0) \nleqslant C_Q(W)$. Thus, in either case we can find $P_0 \leqslant G_{\alpha\beta}$ of order p with $C_Q(P_0) \nleqslant C_Q(W)$. Moreover, $C_Q(P_0) \cap C_Q(W) \neq 1$ since $p \nmid q - 1$. Thus, $C_0(W) \cap C(P_0)$ is $SL(2, \ell)$.

If q is odd then (iii) follows from Lemma 4.4. Suppose q is even. Then $3 \mid |C_0(P_0)|$ implies that $C_0(P_0)^{\Delta(P_0)}$ is not a Suzuki group. Since $C_Q(W) = Z(Q) = \Omega_1(Q)$ and Q has exponent 4 [34, p. 568], $C_0(P_0)^{\Delta(P_0)}$ is a unitary or a Frobenius group. In the latter case, $C_Q(P_0)$ is a Frobenius complement of exponent 4. Then $8 = |C_Q(P_0)| > |C_Q(P_0) \cap C_Q(W)| = 2$ implies that $C_0(P_0)^{\Delta(P_0)}$ is unitary.

LEMMA 15.3. $p \mid q + 1$.

Proof. Otherwise, $3 \neq p \mid q^2 - q + 1$. Choose P_0 as in Lemma 15.2. Let $P_{\alpha\beta}$ be a p-Sylow subgroup of $G_{\alpha\beta}$ containing P_0. Let R be a p-Sylow subgroup of $C_0(P_0)$ normalized by $P_{\alpha\beta}$. Both $P_{\alpha\beta}$ and R are cyclic and $RP_{\alpha\beta}$ is a metacyclic group. By $(*)$ and Lemma 11.1, $RP_{\alpha\beta}$ is a normal subgroup of index p in a p-Sylow subgroup P of G.

Since $1 \lhd R \lhd RP_{\alpha\beta} \lhd P$ with each quotient cyclic, and since $p \geqslant 5$, Lemma 11.2 applies to P. It follows that $N(P)$ has no normal subgroup of index p.

As in Lemma 15.2(iii), $C_Q(P_{\alpha\beta}) \not\leqslant C_Q(W)$, and since $C_0(P_{\alpha\beta}) \leqslant C_0(P_0)$ the group $C_0(P_{\alpha\beta})^{\Delta(P_{\alpha\beta})}$ is unitary. We can find a 3-element $a \in C_0(P_{\alpha\beta})$ normalizing the subgroup $R_1 = \Omega_1(R)$ of $C_0(P_{\alpha\beta})$ and acting nontrivially on R_1. Then $a \in C_0(P_0) \cap N(R_1)$ implies that a normalizes R. Clearly a is fixed-point-free on R. Write $R = \langle r \rangle$ and $r^a = r^i$, where clearly $i \not\equiv \pm 1 (\bmod p)$. By Lemma 15.2(i), $N(P_0) = C(P_0)$, and so $R_1 \not\sim P_0$. Also, $P \leqslant N(P_0)$.

Thus, R_1 is characteristic in $RP_{\alpha\beta}$, and so central in P while P is transitive on the subgroups $\neq R_1$ of R_1P_0 of order p. Consequently, $N(R_1P_0) \leqslant PN(P_0) = PC(P_0)$. Then $N(R_1P_0)/C(R_1P_0)$ is a Frobenius group of order divisible by $3p$, so that a normalizes a p-Sylow group P^* of $N(R_1P_0)$. Let $P^{*g} = P$ with $g \in N(R_1P_0) = P(N(R_1) \cap C(P_0))$. We may assume that $g \in N(R_1) \cap C(P_0)$, and then $R = R^g$, $a^g \in N(R_1) \cap C_0(P^g_{\alpha\beta})$ and a^g is fixed-point-free on R. Since a^g normalizes P we may assume that $P^* = P$ and $g = 1$. Let H be a p-complement of $N(P)$ containing a.

We now claim that $N(P) \leqslant N(R_1P_0)$. If $P_{\alpha\beta} > P_0$, then $R_1P_0 \leqslant \Phi(P) \leqslant RP_{\alpha\beta}$, so that $R_1P_0 = \Omega_1(\Phi(P)) \trianglelefteq N(P)$. Assume now that $P_{\alpha\beta} = P_0$ and R_1P_0 is not normal in $N(P)$. Then RP_0 is not normal in $N(P)$, and as above $P_0 \leqslant P^{(1)}$. Since $R \times P_0 = RP_{\alpha\beta}$ has index p in P it follows that P has class 2. Then $\Omega_1(P) = R_1P_0\langle z \rangle$ with $z \in P - RP_0$ and $\langle z \rangle$ conjugate to P_0. Here $\Omega_1(P)$ is nonabelian of order p^3 and R_1 is its center. Set $\bar{H} = H/C_H(\Omega_1(P))$ and let $\bar{a}$ be the image of a in this group. $\bar{H}$ acts on $\Omega_1(P)/R_1$ and may be regarded as a subgroup of $GL(2, p)$. If $h \in \bar{H}^\#$ normalizes R_1P_0 it centralizes some conjugate of P_0 and hence centralizes R_1P_0/R_1. Thus, $\bar{H}$ contains no nontrivial element of $Z(GL(2, p))$. In particular, $\bar{H}$ contains no Klein group. Since $\bar{H}$ is isomorphic to a subgroup of $PGL(2, p)$ it is cyclic or dihedral (Dickson [9, pp. 285–286]). In particular, $\langle \bar{a} \rangle \trianglelefteq \bar{H}$. Also, a is nontrivial on R_1, so $\bar{a} \neq 1$. Thus, $\bar{H}$ normalizes the centralizer R_1P_0/R_1 of $\bar{a}$ in $\Omega_1(P)/R_1$. Then $\bar{H}$ centralizes R_1P_0/R_1. Since $R_1 \leqslant P^{(1)}$, H centralizes $P_0P^{(1)}/P^{(1)}$. Consequently, $N(P)$ has a normal subgroup of index p, which is a contradiction. Therefore $R_1P_0 \trianglelefteq N(P)$, as claimed.

Now, $N(P) \leqslant N(R_1P_0) \leqslant PN(P_0) = PC(P_0)$ and $N(P) \cap C(P_0)$ normalizes $P \cap C(P_0) = RP_{\alpha\beta}$. Thus, $RP_{\alpha\beta} \trianglelefteq N(P)$.

Note that $H \leqslant PC(P_0)$ implies that we may assume that

$$H \leqslant C(P_0) \cap N(R_1).$$

Then H normalizes $C_0(P_0) \cap N(R_1)$ and hence also R. Since H acts on $RP_{\alpha\beta}/R$ and centralizes RP_0/R, $[H, RP_{\alpha\beta}] \leqslant R$ and $RP_{\alpha\beta} = RL$ with $L = RP_{\alpha\beta} \cap C(H)$. Since a is fixed-point-free on R, L is cyclic and $R \cap L = 1$. Also, $L \geqslant P_0$.

Clearly, $P/\Phi(RP_{\alpha\beta})$ has order p^3. Since $N(P)$ contains no normal subgroup of index p, it follows that $L\Phi(RP_{\alpha\beta}) \leqslant P^{(1)}\Phi(RP_{\alpha\beta})$. Since $P/\Phi(RP_{\alpha\beta})$ is not metacyclic (Huppert [51]) it is extraspecial of exponent p with center $L\Phi(RP_{\alpha\beta})$.

We have $R = \langle r \rangle$, $r^a = r^i$ and $i \not\equiv \pm 1 (\mathrm{mod}\, p)$. Since a centralizes $L\Phi(RP_{\alpha\beta})/\Phi(RP_{\alpha\beta})$, there is an element $z \in P - RP_{\alpha\beta}$ such that $z^a = z^j b$ with $b \in \Phi(RP_{\alpha\beta})$ and $ij \equiv 1 (\mathrm{mod}\, p)$.

Write $R_1 = \langle r_1 \rangle$ and $P_0 = \Omega_1(L) = \langle y \rangle$. Then $[y, z] \in R_1{}^{\#}$. Let $[y, z] = r_1{}^k \neq 1$. Apply a to both sides and obtain

$$r_1^{ik} = [y, z^j b] = [y, z^j] = [y, z]^j = r_1^{jk}.$$

Thus $i \equiv j (\mathrm{mod}\, p)$, so that $i^2 \equiv 1 (\mathrm{mod}\, p)$, whereas $i \not\equiv \pm 1 (\mathrm{mod}\, p)$. This is a contradiction.

LEMMA 15.4. $p = 3$.

Proof. Otherwise, $3 \neq p \mid |W|$ (Lemma 15.2(ii)). Let $P_{\alpha\beta}$ and P_0 be as in Lemma 15.2(iii). Let R be a p-Sylow subgroup of $C_0(W)$ normalized by $P_{\alpha\beta}$. By (*), $P = RP_{\alpha\beta}$ is a p-Sylow subgroup of G. Set $R_1 = \Omega_1(R)$ and $P_1 = \Omega_1(P \cap W)$. Then $R_1 P_1 = \Omega_1(Z(P))$, and, from the structure of P, it follows that $\Omega_1(P) = R_1 P_1 P_0$.

We first show that $N(P)$ contains a normal subgroup of index p. Since $|R : C_R(P_0)| = p$, $|P : C_P(P_0)| = p^2$ or p. Suppose first that $|P : C_P(P_0)| = p^2$. For each $h \in N(P)$, $P_0{}^h \leqslant \Omega_1(P)$ but $P_0{}^h \cap Z(P) = 1$, so that $P_0{}^h = P_0{}^b$ for some $b \in P$. Thus, $N(P) \leqslant PN(P_0) = PC(P_0)$. Set $P_0 = \langle x \rangle$ and consider the image of x under the transfer of $N(P)$ into $P/R(P \cap W)$. If $g \in N(P) \leqslant PC(P_0)$ then for each integer m we have

$$(x^m)^g \equiv x^m \ (\mathrm{mod}\ R(P \cap W)).$$

Thus, $N(P)$ has a normal subgroup of index p in this case. Next suppose that $|P : C_P(P_0)| = p$. Here $P_{\alpha\beta} \leqslant C(P_0)$. Clearly $P_{\alpha\beta}$ contains p subgroups of order p other than P_1, all of which are central in $P_{\alpha\beta}$. Since $N(P_1 P_0)_{\alpha\beta}$ centralizes $P_1 P_0 / P_1$, none of these p subgroups are conjugate in $G_{\alpha\beta}$ and, hence, in G (Lemma 4.3). Thus, the subgroups of $R_1 P_1 P_0$ of order p not in $R_1 P_1$ lie in p classes in G, with each class containing p subgroups and P transitive on each class. Once again it follows that $N(P) \leqslant PC(P_0)$, and $N(P)$ has a normal subgroup of index p.

Since $p \geqslant 5$ and $1 \lhd P \cap W \lhd P_{\alpha\beta} \lhd P$ with each quotient cyclic, as in Lemma 15.3 we can apply Lemma 11.2 to our situation. Then $G/O^p(G) \approx N(P)/O^p(N(P))$, whereas $G = O^{2'}(G)$.

Notation. Let $P_{\alpha\beta}$ be a 3-Sylow subgroup of $G_{\alpha\beta}$, R a 3-Sylow subgroup of $C_0(W)$ normalized by $P_{\alpha\beta}$, and P a 3-Sylow subgroup of G containing

$RP_{\alpha\beta}$. From $(*)$ it follows that $|P : RP_{\alpha\beta}| = 3$, and, hence, $RP_{\alpha\beta} \lhd P$. Set $R_1 = \Omega_1(R) \leqslant C_0(W) \cap C(P_{\alpha\beta}) \leqslant C_0(P_{\alpha\beta})$.

LEMMA 15.5. $3 \mid |W|$.

Proof. Suppose that $3 \nmid |W|$ and set $P_0 = \Omega_1(P_{\alpha\beta})$. Then $R_1P_0 = \Omega_1(RP_{\alpha\beta}) \lhd P$. Since $R \nleqslant N(P_0)$, $R_1 \leqslant Z(P)$, and some element of $P - RP_{\alpha\beta}$ centralizes P_0. Thus, $C(P_0) \nleqslant N(W)$ and hence $C_0(P_0) = PSU(3, \ell)$ (see the proof of Lemma 15.2(iii)). Since $3 \nmid |C_W(P_0)|$, $(\ell + 1)_3 = 3$ and $|R| = (q+1)_3 = (\ell+1)_3(\ell^2 - \ell + 1)_3 = 9$. Moreover, ℓ is not a cube, so that $P_{\alpha\beta} = P_0$. Thus $|P| = 3^4$. Also, $\ell \neq 2$ as $|W| \mid \ell^3 + 1$.

We claim that $C_0(P_0) = PSU(3, \ell)$. For, as $C(P_0)$ does not contain a 3-Sylow subgroup of G (Lemma 11.1), a 3-Sylow subgroup of $C(P_0)$ is abelian. By transfer, $P_0 \nleqslant C_0(P_0)$.

We next show that $R \lhd P$. We have $R_1P_0 \lhd P$ and $C_0(P_0) = PSU(3, \ell)$. From the structure of the group $C_0(P_0) \times P_0$, it follows that there is a 3-Sylow subgroup of $C(P_0)$ having the form $\langle a \rangle \times R_1 \times P_0$; moreover, a can be chosen to normalize the subgroup

$$L = (C_0(W) \cap C(P_0) \cap C(R_1))(W \cap C(P_0)).$$

Here L is an abelian subgroup of order $(\ell+1) \cdot \frac{1}{3}(\ell+1)$. Then $a \in N(C(L))$. Since $\ell \neq 2$, $W \cap C(P_0) \neq 1$ and $C(L) \leqslant C(W)$, so that

$$C(L) = (C(L) \cap C_0(W)W)P_0 \qquad \text{and} \qquad |C(L)| \mid (q+1) \cdot \tfrac{1}{3}(q+1) \cdot 3.$$

Moreover, $C(L)$ is not nilpotent and $C(L)$ has a normal abelian subgroup of index 3, say L_0. Then $a \in N(L_0)$, so that a normalizes the unique 3-Sylow subgroup R of L_0. As $a \in N(P_0)$ we have $R \lhd RP_0\langle a \rangle$. Then $N(RP_0)$ induces a $3'$-group of automorphisms on RP_0/R_1. Consequently $R \lhd P$, as claimed.

As $R \lhd P$, P has class 2. The Hall–Wielandt Theorem (Lemma 11.2) implies that $O^3(N(P)) = N(P)$.

Now $C_0(P_0) = PSU(3, \ell)$ implies that a 3-Sylow subgroup of $C_0(P_0)$ is elementary abelian of order 9. As above, P contains an elementary abelian subgroup of order 27. As P has class 2 and is not of exponent p, $|\Omega_1(P)| = 27$ and $\Phi(P) = R_1$. It follows that $N(P)$ acts on $\Omega_1(P)/\Phi(P)$ as a subgroup of $GL(2, 3)$ of order prime to 3. As $O^p(N(P)) = N(P)$, there must be an element $h \in N(P)$ such that h inverts $\Omega_1(P)/\Phi(P)$. Then h inverts R_1P_0/R_1. But P is transitive on the subgroups $\neq R_1$ of order p contained in R_1P_0. Thus, $h \in PN(P_0) = PC(P_0)$, a contradiction.

We can now complete the proof of Theorem 15.1. Set $P_1 = \Omega_1(P \cap W)$. Then, $R_1P_1 = \Omega_1(Z(RP_{\alpha\beta})) \lhd P$, so P normalizes $C(R_1P_1)$. Since f is now a power of 3, $C(R_1P_1) = (C(R_1) \cap C_0(W))WP_{\alpha\beta}$. Then P normalizes

$C(R_1P_1) \cap C(O_{3'}(C(R_1P_1)))$. If $q \neq 8$, then $q + 1$ is not a power of 3, and $(q+1)/3(\ell+1)$ is not divisible by 3 (Lemma 11.1). Then no element of $P_{\alpha\beta} - P \cap W$ centralizes $O_{3'}(C(R_1P_1))$, and $R(P \cap W)$ is the unique 3-Sylow subgroup of $C(R_1P_1) \cap C(O_{3'}(C(R_1P_1)))$.

We claim that $R(P \cap W) \lhd P$. If this is false, then we must have $q = 8$. Suppose that $P_{\alpha\beta}$ is cyclic. Then $RP_{\alpha\beta}$ is metacyclic of class 2, and $R(P \cap W) \lhd P$ since $R(P \cap W) = \{x \in RP_{\alpha\beta} \mid x^p \in (RP_{\alpha\beta})^{(1)}\}$. Next let $P_{\alpha\beta}$ be noncyclic and choose P_0 as in Lemma 15.2. Here, $C_0(P_0) = PSU(3, 2)$, and we can use the proof of [34, Lemma 60.1] to obtain a contradiction. Thus, $R(P \cap W) \lhd P$.

Note that P_1 and R_1 are not conjugate as R_1 is inverted whereas P_1 is not. Also, P_1 is not normal in P. Thus,

$$\begin{aligned} N(R_1P_1) &= P(N(R_1) \cap N(P_1)) \\ &= P(N(R_1) \cap C(P_1)) \rhd R(P \cap W). \end{aligned}$$

Suppose that R_1P_1 is weakly closed in P. Then by the Hall–Wielandt Theorem, $N(R_1P_1)$ has no normal subgroup of index 3. As $N(R_1P_1)/R(P \cap W)$ has metacyclic 3-Sylow subgroups, $P/R(P \cap W)$ is abelian (Huppert [51]). We have $N(R_1P_1) = \langle P, C(R_1P_1), u\rangle$ with u an involution in $N(R_1) \cap C_0(P_{\alpha\beta})$ inverting R_1. Also, $C(R_1P_1) = (C(R_1) \cap C_0(P_1))WP_{\alpha\beta}$. Thus,

$$[N(R_1P_1), P_{\alpha\beta}] \leqslant (C(R_1) \cap C_0(P_1))W$$

and $N(R_1P_1)$ has a normal subgroup of index 3, which is a contradiction.

Therefore, R_1P_1 is not weakly closed in P. Then, some conjugate P_1^x of P_1 lies in P but not in R_1P_1.

Note that no conjugate P_1^y of P_1 can lie in $C_P(P_1) = RP_{\alpha\beta}$ but not in R_1P_1. For otherwise, $(P_1^y)^\Delta \nleqslant R^\Delta$ and $(RP_{\alpha\beta})^\Delta$ is nonabelian and metacyclic. Therefore there is an element $r \in R$ such that P_1^{yr} is in $P_{\alpha\beta}$ but not in $P \cap W$. Then P_1, P_1^{yr} are conjugate in G but not in $G_{\alpha\beta}$, which contradicts Lemma 4.3.

We thus have $[P_1, P_1^x] \neq 1$. Also $\Delta(P_1) \cap \Delta(P_1^x) = \phi$, since an element of $(P_1^x)^\#$ conjugates P_1 to a subgroup of R_1P_1 having no fixed points on Δ.

Moreover, $C(P_1) \cap C(P_1^x)$ contains no conjugate P_3 of P_1, since otherwise $P_3\langle P_1, P_1^x\rangle = P_3 \times \langle P_1, P_1^x\rangle$ would be conjugate to a subgroup of $C_P(P_1)$, and this is impossible. The proof of Theorem 15.1 will be complete once we prove the following fact.

LEMMA 15.6. *If $x \in G$, $[P_1, P_1^x] \neq 1$ and $\Delta(P_1) \cap \Delta(P_1^x) = \phi$, then there is a conjugate of P_1 lying in $C(P_1) \cap C(P_1^x)$.*

Proof. Let P_1, P_2, P_3, P_4 denote distinct conjugates of P_1. Since $RP_{\alpha\beta}$ contains no conjugate of P_1 outside of R_1P_1, if $[P_1, P_3] = 1$, then $P_3 \leqslant C_0(P_1)P_1$. Moreover, $P_1 \times P_3$ then contains precisely one conjugate

$\neq P_1$, P_3 of P_1. Consequently, $C(P_1)$ contains precisely $2 \cdot \frac{1}{2}q(q-1)$ conjugates $\neq P_1$ of P_1.

Next, note that there are $q^2(q^2 - q + 1)$ conjugates of P_1 in G, q^2 of which lie in G_α. The number of conjugates $P_2 \nleq C(P_1)$ with $\Delta(P_1) \cap \Delta(P_2) = \phi$ is thus $q^2(q^2 - q + 1) - 1 - 2 \cdot \frac{1}{2}q(q-1) - |\Delta|(q^2 - 1) = q(q-1)(q^2 - q - 2)$.

The lemma asserts that whenever $[P_1, P_2] \neq 1$ there is a P_3 in $C(P_1) \cap C(P_2)$. We first show that there is at most one such P_3. For let $P_3, P_4 \leq C(P_1) \cap C(P_2)$. Then we have seen that $\langle P_3, P_4 \rangle \leq C_0(P_1)P_1$. Here, $\langle P_3, P_4 \rangle$ is not a 3-group, as otherwise $\langle P_1, P_3, P_4 \rangle$ would be conjugate to a subgroup of $RP_{\alpha\beta}$, so that $P_1 \leq \langle P_3, P_4 \rangle \leq C(P_2)$, which is not the case. Since $3 \nmid q$, $\langle P_3, P_4 \rangle^{\Delta(P_1)}$ or $\langle P_3, P_4 \rangle^{\Delta(P_2)}$ contains a Klein group (Dickson [9, pp. 285–286]). Now $|\langle P_3, P_4 \rangle|$ is even and $\Delta(P_1) \cap \Delta(P_2) = \phi$, so that q must be odd. Since $C(P_1)^{\Delta(P_1)}$ and $C(P_2)^{\Delta(P_2)}$ have no quaternion subgroups, $\langle P_3, P_4 \rangle$ contains a Klein group $\langle u, v \rangle$. Then $\langle P_1, P_2 \rangle \leq C_0(u)W_u \cap C(v)$ (by the first paragraph), whereas the latter group has a normal abelian 3-Sylow subgroup. This is a contradiction.

We now fix P_1 and count in two ways the ordered pairs (P_2, P_3) with P_1, P_2, P_3 distinct and conjugate, $[P_1, P_2] \neq 1$, $\Delta(P_1) \cap \Delta(P_2) = \phi$, and $[P_1, P_3] = [P_2, P_3] = 1$. On the one hand, we have just seen that each P_2 determines at most one P_3, so that there are at most $q(q-1)(q^2 - q - 2)$ such pairs. On the other hand, each P_3 determines $2 \cdot \frac{1}{2}q(q-1) - 2$ groups P_2 in $C(P_3)$ not in $C(P_1)$. Since there are $2 \cdot \frac{1}{2}q(q-1)$ P_3's, the number of pairs (P_2, P_3) is $(q^2 - q)(q^2 - q - 2)$. It follows that each P_2 does in fact determine a P_3, and this proves the lemma.

As already noted, the preceding lemma provides the contradiction needed to complete the proof of Theorem 15.1.

Remark. The proof of Theorem 15.1 could have been completed without Lemma 15.6 by using an involved fusion argument. The preceding lemma shows that an entirely different kind of structure is available than we have needed before. It is clear that this lemma will hold in other situations: it could have been deduced earlier in Section 15, and was implicitly available, though seemingly not needed, in the work of Suzuki [34, Sections 21–24, and 35], O'Nan [24], and Kantor, O'Nan and Seitz [22] (following their Lemma D.5).

We note that Lemma 15.6 represents the bulk of the construction of a projective plane of order q^2 on which G acts. The same is true of the analogues of this lemma available in the above references.

16. Case II

We now return to the second situation described in Section 12. In this case we wish to prove:

THEOREM 16.1. *G is of Ree type.*

What we will in fact show is that $n = q^3 + 1$, $W = \langle t \rangle \leqslant Z(G_{\alpha\beta})$, $C(t) = \langle t \rangle \times L$ with L a subgroup of $P\Gamma L(2, q)$ containing $PSL(2, q)$ as a subgroup of odd index f, $| G | = (q^3 + 1)q^3(q - 1)f$, and each prime divisor of f divides $q^2 - q + 1$. As in Walter's work [53], it is hard to eliminate field automorphisms of order a power of a prime $p \mid q^2 - q + 1$, and we have not been able to do so even in our permutation group situation. The difficulty is due to the fact that a p-Sylow subgroup of a group of Ree type is not known to be cyclic. However, we remark that Walter's argument is made simpler once one knows $| G |$ and the existence of a character of degree $n - 1 = q^3$ in the principal 2-block [22, Lemma 3.6].

We first show that $t \in Z(G_{\alpha\beta})$. If this is not so let $x \in G_{\alpha\beta} - \langle t \rangle$ with $x^t = x^{-1}$. As in the proof of Lemma 13.2(i), $C_Q(x) \neq 1$. Then $C_0(x)^{\Delta(x)}$ is as in Lemma 4.4. However, $| C_0(x)_{\alpha\beta} |$ is odd and a 2-Sylow subgroup of $C(x)$ has order 4 and can be assumed to be centralized by t. This is impossible for the group $N(\langle x \rangle)^{\Delta(x)}$.

Let $t' = (\alpha\beta) \cdots$ be an involution. Since $\langle t \rangle$ is the 2-Sylow subgroup of $G_{\alpha\beta}$, by Lemma 4.3 t' centralizes W. As in Lemma 13.5 it follows that $n = q^3 + 1$. Then $| G | = \frac{1}{2}(q^3 + 1)q^3(q - 1) | W | f$, where

$$f = | C(t)^{\Delta} : C_0(t)^{\Delta} |.$$

As in Section 13, $C_0(t)_{\alpha\beta}W$ is semiregular on $\Omega - \Delta$. Since $W \leqslant C(t')$, $| W | \mid q + 1$. Also, W is cyclic since $| W |_2 = 2$. If $f = 1$, then $G_{\alpha\beta}$ is cyclic and the theorem follows from [22, Section 5, Case D]. We may thus assume that $f \neq 1$.

As in Section 15, each prime $p \mid f$ divides $q^2 - q + 1$ or $| W |$. Suppose that $p \mid | W |$. If a p-Sylow subgroup of $G_{\alpha\beta}$ is cyclic, then G is not simple by a result of Huppert [51]. Thus, a p-Sylow subgroup of $G_{\alpha\beta}$ is noncyclic and $G_{\alpha\beta}$ contains a subgroup P_0 of order p such that $C_0(P_0)^{\Delta(P_0)}$ is of Ree type. As W is cyclic, it follows that $C_W(P_0) = \langle t \rangle$. This contradicts the supposition that $p \mid | W |$.

Thus, each prime divisor of f divides $q^2 - q + 1$. Since $W \leqslant C(t')$ it follows that $W^{\Delta(t')} \leqslant C_0(t')^{\Delta(t')}$. If $\langle t', u \rangle$ is a 2-Sylow subgroup of $C_0(t)$ then $u^{\Delta(t')}$ centralizes $W^{\Delta(t')}$, which has even order. This is only possible if $| W^{\Delta(t')} | = 2$. Thus $W = \langle t \rangle$.

Clearly, $C(t) = \langle t \rangle \times C_0(t)O(G_{\alpha\beta})$ with $C_0(t)O(G_{\alpha\beta}) \approx (C_0(t)O(G_{\alpha\beta}))^{\Delta}$. At this point it seems to be necessary to invoke Walter's work [53] in order to conclude that $f = 1$.

17. Historical Note

The proof of Theorem 1.1 clearly depends on the classification of Zassenhaus groups [10, 20, 33, and 43]. When $n = |\Omega|$ is odd, results of Suzuki [34], Bender [45] and Shult [30] are required, and these depend in turn upon the Feit–Thompson Theorem [11].

When n is even we have used results of Bender [4], Hering [17], Suzuki [35], O'Nan [24] and Kantor, O'Nan, and Seitz [22], together with the end of Walter [53]. Of these, only Hering's result and that of [22] involve the Gorenstein–Walter Theorem [49]. However, we have used only the even degree case of [22], and then the Gorenstein–Walter Theorem is not needed. Since we have shown how to prove Theorem 1.1 without using the results of Alperin, Brauer and Gorenstein [1, 2], it seems fitting to point out that the Gorenstein–Walter Theorem can also be dispensed with in the proof.

Thus, we will prove the special case of Hering's result [17] which was used in Lemma 4.1(v). The following situation will be considered.

(H) G is a group 2-transitive on a finite set Ω with $n = |\Omega|$ even, some involution t fixes two points α, β, but no involution fixes more than two points.

Using very elementary arguments, Hering [17] observed that (H) implies the following: (a) a 2-Sylow subgroup S of G is dihedral or quasidihedral; (b) if S is quasidihedral then G is 3-transitive; and (c) $G_{\alpha\beta}$ has at most two orbits on $\Omega - \{\alpha, \beta\}$.

Lemma 17.1. *If (H) holds and G_α has a normal subgroup Q regular on $\Omega - \alpha$, then G has a normal subgroup acting on Ω as $PSL(2, q)$ in its usual 2-transitive representation.*

Proof. Clearly, t inverts Q, Q is abelian, and t is the unique involution in $G_{\alpha\beta}$. If $n \equiv 0 \pmod 4$, then t is an odd permutation, so G has a normal subgroup of index 2 all of whose involutions are regular. In this case Bender's result [4] completes the proof.

We now assume that $n \equiv 2 \pmod 4$. Moreover we may assume that no proper normal subgroup of G satisfies (H) in its action on Ω. Then all involutions in G are conjugate and we can choose S so that $S_{\alpha\beta}$ is a 2-Sylow subgroup of $G_{\alpha\beta}$.

If $\gamma \in \Omega - \{\alpha, \beta\}$, then t centralizes the involution u in $G_{\gamma\gamma^t}$, and u interchanges α and β. There are thus $(n-2)/2$ involutions $(\alpha\beta)\cdots$. Consequently, $K = \{x \in G_{\alpha\beta} \mid x^u = x^{-1}\}$ contains precisely $(n-2)/2$ elements.

Next note that Q is a p-group. For otherwise we can write $Q = A \times B$ with $A \neq 1$ and $B \neq 1$ Hall subgroups of Q. Then β^A and β^B are nontrivial imprimitivity classes of G_α having different sizes, so that $G_{\alpha\beta}$ leaves $\beta^A - \beta$ and $\beta^B - \beta$ invariant, which contradicts (c).

We now distinguish between the cases $S_{\alpha\beta}$ cyclic or noncyclic.

(i) Suppose that $S_{\alpha\beta}$ is cyclic. Here $G_{\alpha\beta}$ has a normal 2-complement and $|K \cap O(G_{\alpha\beta})| = (n-2)_{2'}$. By a lemma of Bender [26, Lemma 1.2] and the minimality of G, $G_{\alpha\beta} = \langle K \rangle$. It suffices to show that $G_{\alpha\beta}$ is semiregular on $\Omega - \{\alpha, \beta\}$.

Write $|Q| = p^e$. Here $p^e \equiv 1 (\mathrm{mod}\ 4)$. If p^e is not the square of a Mersenne prime then by [5] there is a prime r dividing $p^e - 1$ but not dividing $p^i - 1$ for $1 \leqslant i < e$. Then $r \mid |K|$ and hence Q is elementary abelian. By an elementary result of Passman [52, Proposition 4.2], G_α can be regarded as a group of certain mappings of the form $x \to ax^\varphi + b$ on $GF(p^e)$, where $a \neq 0$ and b are in $GF(p^e)$ while $\varphi \in \mathrm{aut}\ GF(p^e)$. Since $G_{\alpha\beta} = \langle K \rangle$, $G_{\alpha\beta} = KG_{\alpha\beta}^{(1)}$. If $u = (\alpha\beta) \cdots$ is an involution normalizing $S_{\alpha\beta}$ then $C(u)_{\alpha\beta} \leqslant \langle t \rangle G_{\alpha\beta}^{(1)} \cap \langle t \rangle O(G_{\alpha\beta})$. However, $G_{\alpha\beta}^{(1)}$ is semiregular on $\Omega - \{\alpha, \beta\}$ whereas $O(C(u)_{\alpha\beta})$ fixes each of the fixed points of u. Thus, $C(u)_{\alpha\beta} = \langle t \rangle$ and $G_{\alpha\beta} = K$ is cyclic and semiregular on $\Omega - \{\alpha, \beta\}$, as required.

If $p^e = p^2$ with p a Mersenne prime, then Q is elementary abelian, and $G_{\alpha\beta}$ can be regarded as a subgroup of $GL(2, p)$ containing the central involution of $GL(2, p)$. Since $S_{\alpha\beta}$ is cyclic, $O(G_{\alpha\beta})$ is abelian (Dickson [9, pp. 285–286]), and since we may assume that $(p+1)_2 > 4$ the image of $G_{\alpha\beta}$ in $PGL(2, p)$ is cyclic. Thus, $G_{\alpha\beta}$ is abelian and metacyclic, and as before this implies that $K = G_{\alpha\beta}$ is cyclic. Using Lemma 4.3 we find that $G_{\alpha\beta}$ is semiregular on $\Omega - \{\alpha, \beta\}$.

(ii) Now suppose that $S_{\alpha\beta}$ is noncyclic. Then $S_{\alpha\beta}$ is generalized quaternion and S is quasidihedral. G has no subgroup of index 2. The permutation character θ of odd degree $n - 1$ is in $B_0(2, G)$ [22, Lemma 3.6]. In the notation of Lemma 11.3, $\theta = \chi_i$, $i = 1, 2$, or 3, and $\delta_i = \theta(s) = -1$ since $s \notin G_\gamma$ for any γ. Clearly, $\theta(t) = 1$.

Since $x_2 \equiv -m \equiv -1 (\mathrm{mod}\ 4)$, $\theta \neq \chi_2$. Suppose that $\theta = \chi_1$. Then $1 = \chi_1(t) = \delta_1 m = -m$, whereas $m \equiv 1 (\mathrm{mod}\ 4)$. Thus, $\theta = \chi_3$. Since $\delta_3 = -1$, either δ_1 or δ_2 is 1 and Lemma 11.3(viii) applies. Then $n - 1 = (m^2 - h)(m^2 - h - 1)/h$. However, $n - 1 = |Q|$ is a prime power by (b). Thus, $h = m^2 - h$ or $m^2 - h - 1$. Since m is odd we must have $h = m^2 - h - 1$, so that $m^2 - h = n - 1 \equiv 1 (\mathrm{mod}\ \frac{1}{2}|S|)$. However, $m^2 - h = x_1$ or x_2. If $\delta_1 = -1$ then $1 \equiv x_1 \equiv -2 + m (\mathrm{mod}\ \frac{1}{2}|S|)$, whereas $m \equiv 1 + \frac{1}{4}|S| (\mathrm{mod}\ \frac{1}{2}|S|)$. If $\delta_2 = -1$ then $1 \equiv x_2 \equiv -m (\mathrm{mod}\ \frac{1}{2}|S|)$. These contradictions prove the lemma.

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