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GENERALIZED QUADRANGLES AND TRANSLATION PLANES*

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Abstract

This note will survey some recently noticed relationships between a certain class of generalized quadrangles with parameters q^2, q, and certain classes of translation planes of order q^2. While these relationships seem very mysterious, they provide new constructions for interesting types of geometries of both of these sorts.

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This note will survey some recent work of Payne, Thas, Fisher, Hiramine and myself. It is dedicated to the many hours spent in Dick Bruck's office 20 years ago during which I was introduced to properties of spreads of PG(3,q).

Generalized quadrangles and translation planes are both familiar objects in finite geometry. That they are related at all is somewhat surprising. Relationships will be exhibited, but they are not at all understood and presently seem mysterious. Consequently, a principal theme of this note is the question: what is really going on here?

1. Generalized quadrangles

As usual, q will always denote a prime power > 1. Let $F=GF(q)$ and let $Q=F^2\times F\times F^2$ with multiplication given by

$$(u,c,v)\,(u',c',v') = (u+u',c+c'+v\cdot u',v+v')$$

$(u,u',v,v'\in F^2;\ c,c'\in F)$, where $v\cdot u'$ is the usual dot product. Then Q is a group of order q^5, $Z=Z(Q)=0\times F\times 0$, and $Q/Z \cong F^2\oplus F^2$.

We will be concerned with families $\mathcal{F}$ of q+1 subgroups A of order q^2 such that, for any distinct $A, B, C \in \mathcal{F}$,

$$Q = AZB$$

and

$$AB \cap C = 1.$$

These conditions can be viewed as "independence" restrictions on the members of $\mathfrak{F}$. Each such family $\mathfrak{F}$ determines a generalized quadrangle as follows:

<u>points</u>: symbol $\mathfrak{F}$; cosets AZg; elements $g \in Q$;

<u>lines</u>: symbols $[A]$; cosets Ag;

<u>incidence</u>: $[A]$ I $\mathfrak{F}$, $[A]$ I AZg, and all other incidences are obtained via inclusion.

The resulting generalized quadrangle has parameters q^2, q. For this and a more general construction, see [9]. (N. B. -- The use of this specific group Q of order q^5 is motivated by its having sufficiently many subgroups of order q^2.)

In order to find families $\mathfrak{F}$, we may assume that $\mathfrak{F}$ is parametrized by $F \cup \{\infty\}$ as follows:

$$A(\infty) = 0 \times 0 \times F^2$$

$$A(r) = \{(u, uB_r u^t, uM_r) \mid u \in F^2\}$$

$$\mathfrak{F} = \{A(r) \mid r \in F \cup \{\infty\}\}$$

for 2×2 matrices M_r and B_r satisfying the conditions

$$M_r = B_r + B_r^{\,t}$$

$$B_r = \begin{pmatrix} r & g(r) \\ 0 & f(r) \end{pmatrix}$$

for functions $f,g:F\to F$ for which

(*) the quadratic

$$x^2 - (g(r)-g(s))x + (r-s)(f(r)-f(s))$$

is irreducible for all distinct $r,s\in F$.

Examples.

1. q odd, $g\equiv 0$, $f(r)=-nr$ (n a nonsquare). This produces the classical $PSU(4,q)$ quadrangle. More generally, for even or odd q, when $g(r)=ar$ and $g(r)=br$ with $a,b\in F$ such that x^2-ax+b is irreducible, the resulting quadrangle is classical.

2. q odd, $g\equiv 0$, $f(r)=-nr^\sigma$ with n a nonsquare and $\sigma\in \mathrm{Aut}F$. For $\sigma\neq 1$ the quadrangle is not classical; and $\sigma,\tau\in\mathrm{Aut}F$ produce nonisomorphic quadrangles whenever $\tau\neq\sigma^{\pm 1}$ [10].

3. $g(r)=-r^2$, $f(r)=-r^3/3$, $q\equiv 2 \pmod 3$. This is the generalized quadrangle associated with the $G_2(q)$ generalized hexagon [9]. It is not classical if $q>2$.

4. $g(r)=10r^3$, $f(r)=20r^5$, q odd, $q\equiv\pm 2 \pmod 5$ [10].

$g(r)=r^3$, $f(r)=r^5$, $q=2^{2e+1}$ [13].

These quadrangles are not isomorphic to any of the previous ones if $q>5$, $q\neq 8$ [10,13].

5. There is one further class of examples, the description of which will be postponed until the last section.

We will not discuss properties of the preceding generalized quadrangles. Instead, we will turn to translation planes -- in the guise of spreads of $F^2\oplus F^2$.

Assume that $\mathcal{F}$ is as before, and project it into $Q/Z \cong F^2 \oplus F^2$. This produces a family $\mathcal{F}^*$ of q+1 subspaces of $F^2 \oplus F^2$ each pair of which span $F^2 \oplus F^2$. Can $\mathcal{F}^*$ always be extended to a spread in a "natural" manner? Examples 1 and 2 extend to desarguesian (i. e., regular) spreads. The examples in 3 extend to the spreads defining the Hering-Ott planes [5,12]. No answer is presently known for the remaining examples. However, this question may be too wishy-washy. Therefore, we will now turn to more definite sources of translation planes.

2. Hallian planes

The first type of translation plane we will consider is most easily defined using a quasifield (F^2,$*$):

$$(a,b)*(x,y) = \begin{cases} (ax,bx) & \text{if} \quad y=0 \\ (a,b)\begin{pmatrix} x & y \\ s & t \end{pmatrix} & \begin{array}{l} \text{where } y\neq 0, \\ s = -(x^2/y)+g(1/y)x+f(1/y), \\ t = -x+yg(1/y) \end{array} \end{cases}$$

for functions $f,g:F\rightarrow F$. As noted by Hiramine [6,7], this multipilication defines a quasifield if and only if f and g satisfy (#). (This extends work in [2].) This translation plane admits a group of q F-linear automorphisms fixing a Baer subplane pointwise. (Hiramine [7] conjectures that the converse is true, and presents evidence in this direction.) Moreover, there is an F-linear group of q(q−1) automorphisms with an orbit of length q(q−1) at infinity if, and only if, $f(r)=ar^{-2k+1}$, $g(r)=br^{-k+1}$ for some $a,b \in F$, $k \in \mathbb{Z}$. (Compare §1, Examples 1, 3 and 4.)

We now have a strange correspondence between certain generalized quadrangles with parameters q^2,q and certain translation planes of order q^2. The classical quadrangles correspond to the case $f(r)=ar$, $g(r)=br$, and then $(F^2,*)$ is just a familiar Hall quasifield [4, p. 364] -- hence the title of this section. (Note that in the general case $(F^2,*)$ satisfies the following genralization of a basic property of Hall quasifields: if $u=(x,y)$, $y\neq 0$, then $u^2-yg(1/y)u+yf(1/y)=0$. Of course, the appearance of awkward-looking functions such as $yf(1/y)$ is designed to fit in with (#) but would be irritating to compute with.)

If f and g satisfy (#) then so do f+c and g+d for any $c,d \in F$. While f+c and g+d produce the same quadrangle as f and g do, they yield new translation planes -- e. g., if $f(r)=ar^{-2k+1}$, $g(r)=br^{-k+1}$. Nothing seems to be known about isomorphisms among these planes.

The many-to-one correspondence between these planes and quadrangles makes it seem especially difficult to find a direct geometric relationship between them. We next turn to a more promising situation.

3. Flockian planes

The Klein correspondence behaves as follows:

$W=F^2 \oplus F^2$	↔ Klein quadric in the $O^+(6,q)$ space $V=W\wedge W$
2-space (line)	↔ singular point
intersecting lines	↔ perpendicular points
spread	↔ q^2+1 singular points, no two perpendicular (ovoid)
regulus of q+1 lines	↔ q+1 points of a nonsingular 3-space (conic).

In this section we will describe ovoids that are nice unions of conics.

Let (,) and Q be the bilinear and quadratic form associated with V.

Fix a 4-space C with radical $c=C^{\perp}$ a singular point. The singular points of C form a "cone" of $1+q(q+1)$ points. A flock of C is a set of q nonsingular 3-spaces of C whose union contains all singular points of $C-\{c\}$. In other words, there are q conics producing a partition of the singular points of $C-\{c\}$.

One way to find flocks is to introduce coordinates. There is a basis v_1, v_2, v_3, $v_4=c$ of C such that $Q(v_1)=Q(v_2)=0$, $Q(v_3)=-1$, $(v_1,v_2)=1$ and $(v_1,v_3)=(v_1,v_3)=0$. Any 3-space of C not containing c has an equation of the form

$$rx_1 + sx_2 + tx_3 + x_4 = 0$$

with respect to this basis. A flock consists of q such 3-spaces

$$rx_1 + f(r)x_2 + g(r)x_3 + x_4 = 0,$$

one for each $r \in F$, and q such 3-spaces define a flock if and only if f and g satisfy (#). This observation of Thas [14] produces some type of relationship between certain generalized quadrangles and flocks. This time, slight modifications of f and g produce isomorphic quadrangles and orthogonally equivalent flocks. (For other types of modifications, see below.)

Translation planes arise as follows. If Π is a flock then it is a simple exercise to check that the set of singular points in $\cup\{T^{\perp} | T \in \Pi\}$ is an ovoid. Each such ovoid is a union of q conics, all having a common point (namely, c). They all have two common points iff the resulting translation plane is desarguesian; and this occurs iff f and g are linear iff the associated quadrangle is classical.

Example 3 in §1 produces the planes found by Betten [1] and Walker [15].

Example 2 (with $\sigma \neq 1$) produces ovoids lying in 5-spaces. Therefore, the corresponding spreads are symplectic. However, they are not new: they are precisely the symplectic spreads found in [8, p. 1202] from a different point of view; they are the only known nondesarguesian symplectic spreads of vector spaces of characteristic >3; and the resulting translation planes are among the semifield planes studied by Knuth [11].

The ovoids arising from Example 4 do not seem to have any interesting properties.

Example 5 (Fisher [3, (3.9)]). These flocks are obtained by using a different description of C. Let q be odd, and view C as $GF(q^2)\oplus F^2$, with quadratic form $Q(\alpha,x,y)=\alpha\bar{\alpha}-y^2$. Let $\zeta\in GF(q^2)$ have order q+1. Then the singular points in C are $c=\langle(0,1,0)\rangle$ and $\langle(\zeta^i,x,1)\rangle$ with $0\le i\le q$ and $x\in F$. The flock Π consists of the 3-spaces

(I) $\langle(\zeta^{2i},1,0),(\zeta^{2i+1},1,0),(0,0,1)\rangle$, $0\le 2i<q+1$, and

(II) $\{(\alpha,x,y)|x=ty\}$ where $t\in F$ is such that $t^2(\zeta-\bar{\zeta})^2+4$ is a nonsquare.

This is a more algebraic form of the construction in [3]. Note that each singular point in $\langle(\zeta^{2i},1,0),(\zeta^{2i+1},1,0),(0,0,1)\rangle$ has the form $\langle(\zeta^j,t,1)\rangle$ with $\zeta^j = y(\zeta^{2i+1}-\zeta^{2i}) + t\zeta^{2i}$ for some $y\in F$. Then $\zeta^{j-2i} = y(\zeta-1)+t$. The necessary and sufficient condition on t in order that j and y exist is that

$$1 = y^2(\zeta-1)(\bar{\zeta}-1) + ty(\zeta+\bar{\zeta}-2) + t^2,$$

or, equivalently, that $t^2(\zeta-\bar{\zeta})^2+4$ is a square. This produces (q+1)/2 subspaces (I) and (q−1)/2 subspaces (II). No two members of (II) contain any common singular points; the same is true for a member of (I) and a member of (II). If $\langle(\zeta^j,t,1)\rangle$ lies in both $\langle(\zeta^{2i},1,0),(\zeta^{2i+1},1,0),(0,0,1)\rangle$ and $\langle(\zeta^{2h},1,0),(\zeta^{2h+1},1,0),(0,0,1)\rangle$ then $\zeta^{j-2i}=\zeta^{j-2h}$ while $0\le 2i,2h<q+1$, so that i=h.

This flock Π has associated functions f and g, but these seem to be difficult to work with. If $q>5$ then Π is a new flock (note that all the 3-spaces (I) intersect in a point, while all the 3-spaces (II) intersect in a line).

Further flocks. If Π is a flock, the functions f and g determine a generalized quadrangle as in §1. We coordinatized the corresponding family $\mathcal{F}$ by choosing a specific pair of its members and moving them (using Aut Q) so that they had the form $A(\infty)$ and $A(0)$. Whenever $(\text{Aut } Q)_{\mathcal{F}}$ is not 2-transitive on $\mathcal{F}$ this choice should influence f, g and the corresponding flocks and "flockian" ovoids and planes. It seems very likely that this will produce large numbers of new planes. Consequently, generalized quadrangles should provide a new type of replacement process for transforming one spread into several new ones, thereby yielding many new translation planes.

Nevertheless, the principal question is: where are the planes "inside" the quadrangles, or vice versa?

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