Note on permutation groups of degree $n \leq 10^7$

At the request of J. J. Cannon, I have upgraded the algorithms SIMPLY and COMPFY in [Ka] to the range $n \leq 10^7$. The required modifications are given below. Afterwards there are remarks concerning these modifications.

Table 1 – addendum

n^*	y
16	$16!/2$
17	$\|PSL(2,16)\|$; $17!/2$
18	$\|PSL(2,17)\|$; $18!/2$
19	$19!/2$
20	$\|PSL(2,19)\|$; $20!/2$
21	$7!/2$; $\|PSL(3,4)\|$; $21!/2$
22	$\|M_{22}\|$; $22!/2$
23	$\|M_{23}\|$; $23!/2$
24	$\|PSL(2,23)\|$; $\|M_{24}\|$; $24!/2$
25	$25!/2$

Table 2

m	u
5	$5!/2$
6	$5!/2$; $6!/2$
7	$\|PSL(3,2)\|$; $7!/2$
8	$\|PSL(2,7)\|$; $\|AGL(3,2)\| = 8 \cdot 168$; $8!/2$
9	$\|PSL(2,8)\|$; $9!/2$
10	60; $2^4 \cdot 60$; 360; $10!/2$

SIMPLY

The only changes are the above tables, and the following modification of part of Step 7:

$|G| = tu \cdot y^m$ where y is as in Table 1, m and u are as in Table 2, and either $t = 2^i | 2^m$, $t = 2^{2i} | 2^{2m}$ when $n^* = 10$ and $y = 360$, $t = 3^i | 3^m$ when $y = |PSL(2,8)|$, or $t = 2^i 3^j | 6^m$ when $n^* = 21$ and $y = |PSL(3,4)|$.

COMPFY

Steps 1-5 are unchanged.

6. WLOG $n = n^{*m}$ for some $n^* \geq 5$ and $m \geq 5$. Find such an n^* and m.

(Here $5 \leq m \leq 10$ and $5 \leq n \leq 25$ since $n \leq 10^7$. The only new cases not covered by the published version of COMPFY are $n = 5^{10}, 6^8, 7^8, 8^7, 9^7, 10^7, 11^6, 12^6, 13^6, 14^6, 15^6$, and n^{*5} for $16 \leq n^* \leq 25$. Note that n uniquely determines n^* and m except when $n = 5^{10} = 25^5$.)

7. If $|G|$ *cannot* be written in the form

$|G| = tu \cdot y^m$ where y is as in Table 1, m and u are as in Table 2, and either

7.1. $t = 2^i | 2^m$,

7.2. $t = 2^{2i} | 2^{2m}$ when $(n^*, y) = (10, 360)$ or $(17, |PSL(2,16)|)$,

7.3. $t = 3^i | 3^m$ when $(n^*, y) = (9, |PSL(2,8)|)$, or

7.4. $t = 2^i 3^j | 6^m$ when $(n^*, y) = (21, |PSL(3,4)|)$,

then either

n is not a prime power, in which case output the simple group G, or

n is a prime power, in which case proceed as in 5.2.

8. WLOG $|G|$ can be written in the form $|G| = tu \cdot y^m$ given in 7. (This does not uniquely determine u: both $2^3 \cdot 168$ and $2^4 \cdot 60$ can arise as tu in more than one way.)

If the pair (n^*, y) is not $(8, 168)$, then output as follows:

m copies of a simple group of degree n^* and order y (the apparent ambiguities about this statement are removed by noticing that the degree and order uniquely determine each of the groups in question),

in 7.1, i copies of Z_2,

in 7.2, $2i$ copies of Z_2,

in 7.3, i copies of Z_3,

in 7.4, i copies of Z_2 and j copies of Z_3;

and a simple group of order u — unless either

$u = 8 \cdot 168$, in which case output instead $PSL(3,2)$ and 3 copies of Z_2, or

$u = 16 \cdot 60$, in which case output instead A_5 and 4 copies of Z_2.

(Note that the latter possibilities for u are actually subsumed in earlier ones unless t is large.)

9. Same as in [Ka].

Remarks. As already noted, there are some minor ambiguities concerning the formula $|G| = tu \cdot y^m$: it may not uniquely determine the integers t and u. This was also the case in [Ka], due to an oversight; but the outputs for SIMPLY ande COMPFY are in no way affected by this. (However, remark (iv) on p. 523 is not correct in this case, without a bit more care. COMPSER also needs a minor change due to this error.)

The proof of correctness proceeds as in [Ka]. However, there is one annoying case that requires some additional work. This occurs when $n^* = 8$, $y = 8!/2$ and $m = 5$. The numerical approach on p. 523 for showing that there is no earns, is not sufficient: it only shows that, if there is an earns then a 1-point stabilizer contains Sylow 5- and 7-subgroups of $SL(21, 2)$. Instead, I wound up beating this possibility to death using information about irreducible subgroups of $SL(21, 2)$.

[Ka] W. M. Kantor, Finding composition factors of permutation groups of degree $n \leq 10^6$. J. Symbolic Computation 12 (1991) 517-526.

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