2-Transitive Collineation Groups of Finite Projective and Affine Spaces, by W. Kantor (U. of Wisconsin)

A <u>symmetric design</u> is a finite incidence structure of v points and blocks, $k \leq v + 2$ points on each block and dually, and λ points common to every pair of blocks and dually. Here $\lambda(v-1) = k(k-1)$, so that $k|(v-1)$ iff $(k,\lambda) = 1$. We shall consider the following

<u>Problem</u>. Let Γ be a 2-transitive automorphism group of a symmetric design $\mathcal{D}$. (2-transitivity on points and on blocks are equivalent.)

(I) What is $\mathcal{D}$?

(II) Given $\mathcal{D}$, what is Γ ?

We shall present some results related to (I) and (II), together with some results of a less specific nature. In [4] we have proved several results concerning (I), under additional hypotheses on Γ and the parameters v, k and λ. The following is a typical example.

<u>Theorem 1</u>. Let Γ be a 2-transitive automorphism group of a symmetric design $\mathcal{D}$ such that, for each block B, Γ_B is 2-transitive on both B and $\mathcal{C}B$. If $k|(v-1)$, and either $k - \lambda$ is a prime power or $(k - \lambda, 2\frac{(v-1)}{k} - 1) = 1$, then $\mathcal{D}$ is a projective space on the unique design with $v = 11$, $k = 5$, $\lambda = 2$.

Even with these strong transitivity assumptions it seems difficult to prove that, in addition, Γ contains the little projective group or is A_7. This is a special case of (II). Wagner [5] has studied 2-transitive collineation groups of finite projective spaces, and was able to completely take care of the cases of dimensions 3 and 4. In the general case he was, however, able to show that such a collineation group has a simple, normal 2-transitive subgroup. This can be generalized in 2 directions.

Theorem 2. Let $\mathcal{D}$ be a symmetric design with $k|(v-1)$.

i) A 2-transitive automorphism group of $\mathcal{D}$ has a simple, normal 2-transitive subgroup.

ii) An automorphism group of $\mathcal{D}$ fixing a block B, faithful on B, 2-transitive on B, and transitive on the blocks $\neq B$ has a simple, normal subgroup with these same properties.

The following result, concerning affine spaces, is required in the proof of Theorem 2.

Theorem 3. Let $\Gamma = \Delta \prod$, $\Delta \cap \prod = 1$, be a 2-transitive collineation group of a finite affine space or translation plane $\mathfrak{A}$ containing the full translation group $\prod$ of $\mathfrak{A}$, $|\prod| = v$. If Δ has 2 point-orbits of lengths $k > 1$ and $v - k$, then $v = 2^{2e}$ and $k = 2^{2e-1} \pm 2^{e-1}$ for some $e \geq 2$.

The proof of Theorem 1 is rather long and involved, depending upon a lot of computation together with some deep results on 4-transitive groups. The proofs of Theorems 2 and 3 are, however, of an elementary and geometric nature and will now be sketched.

<u>Proof of Theorem 2</u>. If Π is a minimal normal subgroup of Γ then either Π is non-regular and simple or regular and elementary abelian.

(1) Γ_B is transitive on B and $\mathcal{C}B$

(2) Γ_B is flag-transitive on the design $\mathcal{D}_B$ of points off of B and blocks $\neq B$

(3) Γ_B is a primitive on $\mathcal{C}B$

Here (1) follows from the Dembowski-Hughes-Parker Theorem, and (2) follows from (1) and $(k,v-k) = 1$. (3) is proved from (2) and $(k,\lambda) = 1$ using an arguement of Higman and McLaughlin [2]

<u>Case 1</u>. Π is simple and non-regular. Since $1 \neq \Pi_B \leq \Gamma_B$, Π_B is transitive on $\mathcal{C}B$ by (3) and splits B into orbits all of the same length $k_1 | k$. Then $(k_1, v-k) = 1$ implies that Γ_{pB} is transitive on $\mathcal{C}B$ for every p on B, from which it follows that Γ_p is transitive on points $\neq p$.

<u>Case 2</u>. Π is regular and elementary abelian. Then Γ may be regarded as a 2-transitive collineation group of an affine space over the prime field, and Theorem 3 with $\Delta = \Gamma_B$ yields a contradiction to $k | (v-1)$. (It is easy to see that $|\Pi|$ is not prime.)

Before proving Theorem 3 we note a consequence of (3). If Γ_B is not faithful on B then (3) implies that the pointwise stablizer of B is transitive on $\mathcal{C}B$. It is then easy to apply the dual of the Dembowski-Wagner Theorem [1] to deduce that $\mathcal{D}$ is a projective space (in which case Γ contains the little projective group). The same result has been obtained independently by Ito [3] without, however, the numerical restriction $k|(v-1)$.

Theorem 3 depends on 3 lemmas.

Lemma 1 (Mann). If $\mathcal{D}$ is a symmetric design with v and $k - \lambda$ powers of the same prime, then

$$v = 2^{2e} \quad \text{and} \quad k = 2^{2e-1} \pm 2^{e-1}$$

for some e.

Lemma 2. If Γ is a 2-transitive collineation group of a finite affine space, then Γ is hyperplane transitive and, for each hyperplane H, Γ_H is transitive on both H and $\mathcal{C}H$.

This follows from a simple application of the Dembowski-Hughes-Parker Theorem.

Lemma 3. Let $\mathcal{O} = AG(d,p)$ with p prime, Γ a group and Π a normal subgroup of Γ. If Γ admits a faithful representation as a hyperplane-transitive collineation group of $\mathcal{O}$ such that Π is the full translation group, then Γ_H

is uniquely determined by the triple $(\mathcal{A},\Gamma,\Sigma)$ up to conjugacy in Γ.

Proof of Theorem 3. We may assume that $\mathcal{A} = AG(d,p)$, p prime. Let B be the orbit of Δ of length k. Let $\mathcal{D}$ be the incidence structure whose points are the points of $\mathcal{A}$ and whose blocks are the distinct sets B^{γ}, $\gamma \in \Gamma$. Then $\mathcal{D}$ is a symmetric design, and Γ is a 2-transitive automorphism group of $\mathcal{D}$. Since Π is transitive and regular on blocks, and Π is a normal elementary abelian subgroup of Γ, Γ may be regarded as a collineation group of an affine space $\mathcal{A}^{\#} \approx \mathcal{A}$ whose "points" are the blocks of $\mathcal{D}$. We now have 3 interrelated geometries to work with, $\mathcal{A}$, $\mathcal{A}^{\#}$ and $\mathcal{D}$.

By Lemmas 2 and 3, if H is a hyperplane of $\mathcal{A}$ there is a hyperplane $H^{\#}$ of $\mathcal{A}^{\#}$ such that

$$\Gamma_H = \Gamma_{H^{\#}},$$

and Γ has two orbits of pairs (B,H) with B a block of $\mathcal{D}$ and H a hyperplane of $\mathcal{A}$. It follows that Γ_B has an orbit B^b of hyperplanes of $\mathcal{A}$ with the property

$$B \in H^{\#} \Longleftrightarrow H \in B^b.$$

It is now possible to apply standard numerical methods to the orbits of $\Gamma_H = \Gamma_{H^{\#}}$ on $\mathcal{D}$ and those of Γ_B on $\mathcal{A}$ in order to obtain the hypotheses of Lemma 1 and thus prove Theorem 3.

Besides the designs listed in Theorem 1, there is a third family of 2-transitive symmetric designs which we shall now describe. These satisfy the hypotheses of Theorem 3.

Let $\mathcal{Q}$ be an affine translation plane of order $q = 2^e > 2$ with line at infinity ℓ_∞. Let Π be the translation group of $\mathcal{Q}$. Suppose that $\mathcal{O}$ is a line oval of the projective plane $\mathcal{Q} \cup \ell_\infty$ whose knot is ℓ_∞. Let B be the set of points of $\mathcal{Q}$ lying on some line of $\mathcal{O}$. Then B is a difference set in Π having the parameters $v = |\Pi| = q^2$ and $k = q(q+1)/2$. In particular, if $\mathcal{O}$ is a line conic then $\Gamma = \Delta\Pi$ is a 2-transitive automorphism group of the symmetric design determined by B; here Δ is any group such that $PSL(2,q) \leq \Delta \leq P\Gamma L(2,q)$.