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On a Class of Jordan Groups

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To Helmut Wielandt, on his sixtieth birthday, 19 December, 1970

The purpose of this note is to prove the following two results.

Theorem 1. *Let Γ be a finite group 2-transitive on a set S of v points. Let $B \subset S$, where $|B| = k < v-1$ and Γ is not k-transitive. Suppose that the global stabilizer Γ_B of B has a normal subgroup fixing B pointwise and sharply transitive on $S-B$. Then Γ acts on S as one of the following groups:*

(i) *A subgroup of $P\Gamma L(d, q)$ containing $PSL(d, q)$, in its usual 2-transitive representation;*

(ii) *A collineation group of $AG(d, q)$, containing the group $ASL(d, q)$ of collineations generated by elations, in its usual 2-transitive representation on $AG(d, q)$;*

(iii) *A_7 in its 2-transitive representation of degree 15;*

(iv) *There is a regular normal subgroup, and if $x \in S$ then Γ_x acts on $S-\{x\}$ as the group mentioned in* (iii)*; or*

(v) *M_{22}, Aut M_{22}, M_{23} or M_{24} in its usual permutation representation.*

Theorem 2. *Let Γ, S and B satisfy the hypotheses of the first two sentences of Theorem 1. Suppose that Γ_B has a normal nilpotent subgroup fixing B pointwise and transitive on $S-B$. Then either conclusion* (i) *or* (v) *of Theorem 1 holds or conclusion* (ii) *holds and $q=2$.*

We remark that the converses of these theorems hold except that we must have $d \geqq 3$ in (i), while in (ii) $d \geqq 2$ if $q>2$ and $d \geqq 3$ if $q=2$.

Theorem 1 extends results of Ito [6] and the author [7, 8] concerning Jordan groups. The particular case in which the given normal subgroup of Γ_B is abelian was treated in [8] and will be used here.

In [7] a discussion of Jordan groups and some techniques for handling them can be found. Only a few simple facts will be assumed from [7] in the present paper.

The proof of Theorem 1 leans heavily on recent results of Shult [11, 12] and Hering, Kantor and Seitz [3] concerning 2-transitive groups. Moreover, many of the ideas in our proof are easily traced to [3]. We refer the reader to [11] and

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[3] for those properties of $PSL(2, q)$, $Sz(q)$, $PSU(3, q)$ and groups of Ree type which will be needed here.

Theorem 2 is a simple consequence of Theorem 1. Although a proof can be given using a result of Kantor and Seitz [10], we have chosen a more direct geometric proof.

Suppose that S, B and Γ satisfy the hypotheses of Theorem 1 or 2. Let $\Pi \leqq \Gamma$ and $T \subset S$. Then $\Pi(T)$ will denote the pointwise stabilizer of T, Π_T the global stabilizer of T, and Π_T^T the group induced on T by Π_T.

A subset of S of the form B^α, $\alpha \in \Gamma$, will be called a block. The points of S, together with these blocks, form a design $\mathscr{D}$. For purposes of induction, it will be convenient to prove slightly stronger forms of Theorems 1 and 2 concerning (non-degenerate) designs.

Theorem 1'. *Let Γ be an automorphism group of a design $\mathscr{D}$ 2-transitive on points and transitive on blocks. Let B be a block, and suppose that Γ_B has a normal subgroup $\Pi \leqq \Gamma(B)$ sharply transitive on the set $S-B$ of points not in B. Then one of the following holds:*

(i) *$\mathscr{D}$ consists of the points and hyperplanes of $PG(d, q)$ and $\Gamma \geqq PSL(d+1, q)$;*

(ii) *$\mathscr{D}$ consists of the points and hyperplanes of $AG(d, q)$ and $\Gamma \geqq ASL(d, q)$;*

(iii) *$\mathscr{D}$ consists of the points and lines of $PG(3, 2)$, and Γ is A_7;*

(iv) *$\mathscr{D}$ consists of the points and planes of $AG(4, 2)$, Γ contains the translation group, and if $x \in S$ then Γ_x is A_7; or*

(v) *$\mathscr{D}$ is the design associated with M_v, $v=22$, 23, or 24, and Γ is M_{22},* Aut M_{22}, M_{23} *or* M_{24}.

Proof. Let $\mathscr{D}$ and Γ yield a counterexample with v minimal. Then Γ is not 3-transitive ([5]; [7], p. 481).

We shall use the following additional notation. If $\gamma \in \Gamma$ we write $\Pi(B^\gamma) = \Pi^\gamma$. Lines and planes are defined in [2], pp. 65–66 and [7], pp. 472–473. If $x \in S$ then $\Gamma(x)$ denotes the linewise stabilizer of x, and $\hat{\Gamma}_x$ denotes the permutation group induced by Γ_x on the lines on x. If $\Phi \subseteq \Gamma$, then $F(\Phi)$ is the set of fixed points of Φ.

Fix points $x \in B$ and $y \notin B$.

Lemma 1. *Let $X \neq B$ be a block on y such that $|B \cap X|$ is as large as possible. Then $\Pi(X)_B$ is sharply transitive on $B - B \cap X$, and Γ_{yB}^B is 2-transitive on B.*

Proof. Let $x_1, x_2 \in B - B \cap X$. Then $x_2 = x_1^\gamma$, $\gamma \in \Pi(B)$, and $B \cap B^\gamma$ contains $B \cap X$ and x_2. Thus, $B = B^\gamma$, proving the first statement. For the second, see [17], p. 36, or [7], (3.4).

Lemma 2. *Let $\Phi \subset \Gamma$, and suppose that Φ fixes B and $F(\Phi) \nsubseteq B$. Then $C_\Pi(\Phi)$ is transitive on $F(\Phi) - B \cap F(\Phi)$.*

Proof. Let $x_1, x_2 \in F(\Phi) - B \cap F(\Phi)$. Then $x_2 = x_1^\gamma$, $\gamma \in \Pi$. If $\varphi \in \Phi$, then $\Pi^\varphi = \Pi$ and $x_1^{\varphi\gamma} = x_1^\gamma = x_1^{\gamma\varphi}$, so that $\varphi\gamma\varphi^{-1}\gamma^{-1} \in \Pi_{x_1} = 1$. Thus, $\gamma \in C_\Pi(\Phi)$.

We next note that, since Γ is not 3-transitive and Γ is transitive on ordered triples of non-collinear points, lines have more than 2 points ([7], (3.6)).

Lemma 3. *The points and lines of $\mathcal{D}$ do not form $PG(d,q)$ or $AG(d,q)$.*

Proof. Otherwise, B is a subspace. Set $e = \dim B$. If $e=1$ then Γ is transitive on i-spaces for $i \leqq 2$. If $e>1$, by Lemma 1 and the minimality of v, either Γ_{yB}^{B} contains $PSL(e+1,q)$ or $ASL(e,q)$, or we have $PG(d,2)$, $e=3$, and Γ_{yB}^{B} is A_7.

(i) Suppose that Γ_{yB}^{B} is not A_7. Then Γ is transitive on i-spaces for $i \leqq e+1$ ([7], (3.10)). Here we allow $e=1$. Clearly, $e<d-1$. Let $q=p^a$ with p prime.

First suppose that we have $PG(d,q)$ and $p \neq 2$. Let $\sigma \in \Gamma_y$ be an involution. Then σ fixes pointwise two subspaces T_1 and T_2 which span $PG(d,q)$. Let $\dim T_1 = j$ and $\dim T_2 = d-1-j$. If j or $d-1-j$ is 0, then $\Gamma \geqq PSL(d+1,q)$ (Higman [4]; Wagner [15]), so that since $d>e+1 \geqq 2$ we can find another involution in Γ which is not a perspectivity. So suppose that $j \neq 0 \neq d-1-j$. Since $e \leqq d-2$ and Γ is transitive on e-spaces, there is a block X such that $X \nsupseteq T_1$, $X \nsupseteq T_2$, and $X \cap T_1$ and $X \cap T_2$ span X. Then σ fixes X. By Lemma 2, $C_{\Pi(X)}(\sigma)$ has an element fixing $X \cap T_1$ and $X \cap T_2$ and moving a point of $T_1 - X \cap T_1$ to a point of $T_2 - X \cap T_2$. This is impossible.

Thus, $|\Pi| = (q^{d+1} - q^{e+1})/(q-1)$ or $q^d - q^e$ is even. Each involution in Π fixes each $(e+1)$-space $R \supset B$. Thus, the subgroup of Π generated by the involutions in Π fixes R, is faithful on R, and induces a group of perspectivities on R with axis B. Since Π_R is transitive on $R-B$, Π has a normal elementary abelian Sylow p-subgroup K of order q^{e+1} or q^e. K fixes each subspace containing B.

Let $1 \neq \alpha \in K$. Then B is the set of fixed points of α. Let F be an $(e-1)$-space in B. The e-spaces containing F form a $PG(d-e,q)$. α induces an elation of $PG(d-e,q)$ with center B. In view of the action of K on R, we may assume that α moves some e-space in R containing F. Also, $\Gamma(F)$ is 2-transitive on $PG(d-e,q)$. Thus, $\Gamma(F)$ induces a collineation group containing $PSL(d-e+1,q)$ (Higman [4], Wagner [15]).

Let $M \subseteq F$ be an m-space, where $0 \leqq m \leqq e-1$. By induction, except possibly when $q=2$ and $d-m-1=3$, $\Gamma(M)$ induces a group on $PG(d-m-1,q)$ containing $PSL(d-m,q)$ (Higman [4], Wagner [15]). Since Γ is not 3-transitive, either Γ is A_7 acting on $PG(3,2)$, which is not the case, or Γ contains $PSL(d+1,q)$ or $ASL(d,q)$ ([7], § 5).

Suppose that Γ contains $PSL(d+1,q)$, $q>2$. Consider the collineation γ represented by the diagonal matrix with i entries f, i entries f^{-1} and 0 or 1 entry 1, where $0, 1 \neq f \in GF(q)$ and $i = [\frac{1}{2}(d+1)] \geqq 2$. Precisely as for $PSL(d+1,q)$, q odd, we obtain a contradiction.

If Γ contains $ASL(d,q)$, $q>2$, let α be an elation whose axis H is a hyperplane of $AG(d,q)$ parallel to but not containing B. By Lemma 2, $C_{\Pi}(\alpha)$ is transitive on H, whereas $q^{d-1} \nmid |\Pi|$.

Suppose that Γ is $PSL(d+1,2)$. Let Σ be the translation group of $AG(d+1,2)$. The group $\Gamma\Sigma = ASL(d+1,2)$ then satisfies the conditions of Theorem 1′, with B now an affine $(e+1)$-space, not a hyperplane. We now obtain the same contradiction as in the preceding paragraph.

(ii) As in the preceding paragraph, we now need only consider the following situation: Γ is a collineation group of $AG(d,2)$, containing the translation group, 3-transitive on points, satisfying the hypotheses of Theorem 1′ with B a 4-space, and such that Γ_{yB}^{B} is A_7. Clearly, each 4-space $\| B$ is a block. Γ is transitive on planes.

Two blocks meet in at most 4 points. Thus, there are $\frac{(2^d-2^2)(2^d-2)(2^d-1)}{(2^4-2^2)(2^4-2)(2^4-1)}$ blocks on x, of which $\frac{(2^d-2)(2^d-2^2)}{(2^4-2)(2^4-2^2)}$ are on x and y and $(2^d-2^2)/(2^4-2^2)$ contain a given plane. Since $|\Pi|=2^d-2^4$ is even, we have $d \leqq 2e=8$. It is now easy to see that $d=8$.

As in (i), the group K generated by the involutions in Π is elementary abelian of order 2^4 and fixes each 5-space $R \supset B$. In view of its action on R, K fixes each 4-space $\| B$.

We claim that $\Gamma(T)=1$ for each proper subspace $T \supset B$. For let $\Gamma(T) \neq 1$, where dim T is chosen maximal. Let B^* be a block $\| B$ such that $B^* \cap T = \emptyset$. The Sylow 2-subgroup K^* of $\Pi(B^*)$ fixes each block $\| B^*$ contained in T, and hence fixes T. As K^* is elementary abelian of order >2 and normalizes $\Gamma(T)$, $K^* \Gamma(T)$ is not a Frobenius group with complement K^*. Then some element $\gamma \neq 1$ of $\Gamma(T)$ centralizes an element $\neq 1$ of K^*, and hence fixes B^*. Clearly $F(\gamma) \supseteq T$, and $F(\gamma)$ is a subspace. By Lemma 2, $C_{\Pi(B^*)}(\gamma)$ is transitive on $F(\gamma)$, whereas $|F(\gamma)| \nmid |\Pi|$.

Thus, $|\Gamma_x|=(2^8-1)(2^8-2)(2^8-2^2)(2^8-2^4)$. Let Σ be a Sylow 127-subgroup of Γ_x. Then Σ fixes a unique point $\neq x$, say y. $N(\Sigma)$ faithfully induces a group on $PG(6,2)$. Thus, $|N(\Sigma)|=127c$, $c=1$ or 7 (Dembowski [2], p. 35). It follows that

$$|\Gamma_x : N(\Sigma)_x| \equiv (2-1)\,2(2-4)(2-16)/c \not\equiv 1 \pmod{127},$$

contradicting Sylow's theorem.

Lemma 4. (i) *Blocks are lines.*

(ii) *There are* $r=(v-1)/(k-1)$ *lines on* x *and* $r \geqq k+2$.

Proof. (i) Otherwise, let E be a plane meeting B in a line. Then Π_E is sharply transitive on $E - B \cap E$. It follows that Γ_E^E satisfies the hypotheses of Theorem 1′. Thus, any 3 non-collinear points are contained in a unique $PG(2,q)$ or $AG(2,q)$. By [7], Theorem 6.5, or the Veblen and Young axioms [14] and results of Bruck ([2], pp. 100–101) and Buekenhout [1], the points and lines of $\mathscr{D}$ form $PG(d,q)$ or $AG(d,q)$, contradicting Lemma 3.

(ii) This is immediate by (i).

We note that Lemmas 1 and 4 imply that Γ_{yB}^{B} satisfies the hypotheses of results of Shult [12] and Hering, Kantor and Seitz [3].

Lemma 5. (i) $\Pi \cap \Gamma(x)=1$.

(ii) *Π is nonabelian.*

(iii) *$\Gamma(x)$ acts regularly on $S-\{x\}$.*

Proof. (i), (ii). [8], Theorem 1 and Corollary 1.

(iii) By (i), $[\Pi, \Gamma(x)] \leqq \Pi \cap \Gamma(x) = 1$. Since $\langle \Pi(X) | x \in X \rangle$ is transitive on $S - \{x\}$, (iii) follows.

We define a quadrangle to be a set of 4 points, no 3 collinear. As usual, xy denotes the line joining x and y.

Lemma 6. *Let $\Phi \subset \Gamma$ fix a quadrangle pointwise. Set $C_0(\Phi) = \langle C_{\Pi(X)}(\Phi) | \Phi$ fixes at least 2 points of $X \rangle$. Then either*

(i) *There are 3 collinear points fixed by Φ, and $C_0(\Phi)^{F(\Phi)}$ satisfies the hypotheses of Theorem 1'; or*

(ii) *No 3 fixed points of Φ are collinear, and $C_0(\Phi)^{F(\Phi)}$ is 3-transitive.*

Proof. Let x, y, z be non-collinear points in $F(\Phi)$. By hypothesis, we can find $w \in F(\Phi)$ such that $y, z \notin xw$. By Lemma 2, $C_{\Pi(xw)}(\Phi)$ has an element moving y to z. Thus, $C_0(\Phi)^{F(\Phi)}$ is 2-transitive. Lemma 2 now implies that (i) or (ii) holds.

Lemma 7. *Suppose that $\Phi \subset \Gamma$ fixes a quadrangle pointwise. Let $\Phi, \Phi^\gamma \subseteq \Gamma_{xyz}$, where x, y, z are non-collinear. Then Φ and Φ^γ are conjugate in Γ_{xyz}.*

Proof. As $x, y, z, x^\gamma, y^\gamma, z^\gamma \in F(\Phi^\gamma)$, by Lemma 6 we can find $\delta \in C_0(\Phi^\gamma)$ such that $x^{\gamma\delta} = x$, $y^{\gamma\delta} = y$, $z^{\gamma\delta} = z$. Then $\Phi^{\gamma\delta} = \Phi^\gamma$ and $\gamma\delta \in \Gamma_{xyz}$.

Lemma 8. *Suppose that $\Gamma(B)_y \neq 1$. Then*

(i) *$\Gamma(B)^{S-B}$ is a Frobenius group; and*

(ii) *Π is nilpotent.*

Proof. (i) If $1 \neq \gamma \in \Gamma(B)_y$ fixes a point $\neq y$ of $S - B$, then, as $k = |B| > 2$, γ fixes a quadrangle pointwise. By Lemma 6(i) and the minimality of v, $C_0(\gamma)^{F(\gamma)}$ is a known group: $PSL(3, q)$, $ASL(2, q)$ or A_7. Note that $B \subset F(\gamma)$. As Γ is transitive on ordered triples of non-collinear points each such triple belongs to a unique $PG(2, q)$ or $AG(2, q)$, where $k = q + 1$ or q, respectively. By the Veblen and Young axioms [14] and results of Bruck ([2], pp. 100–101) and Buekenhout [1], $\mathscr{D}$ consists of the points and lines of a projective or affine space. This contradicts Lemma 3.

(ii) This follows from (i) and a result of Thompson [13].

Lemma 9. *$v - k$ is even.*

Proof. Suppose that $v - k$ is odd. By Lemma 4(ii), k is even and v is odd. By Lemmas 8(i) and 5(ii), $\Gamma(B)_y$ has odd order. By Lemma 1 and [3], there is a Klein group $\langle \sigma, \sigma' \rangle$ in Γ_{yB} such that σ, σ' and $\sigma\sigma'$ are conjugate in Γ_{yB}.

By Lemmas 8(i) and 5(ii), if σ fixes a line B_0 and a point $\notin B_0$, then σ fixes additional points not in B_0. In particular, σ fixes a point $y_1 \notin B$, $y_1 \neq y$. As $v - k$ is odd, σ fixes a point $y_2 \notin yy_1$, and then σ fixes at least 3 points $\notin yy_1$ and at least 3 points $\notin yy_2$. Thus, σ fixes a quadrangle pointwise. Also, $4 < |F(\sigma)| \equiv 1 \pmod 2$, and $|F(\sigma) - B \cap F(\sigma)| = |C_\Pi(\sigma)|$ is odd (Lemma 2). We can apply Lemma 6 to σ.

Suppose that $F(\sigma)$ contains 3 non-collinear points. By the minimality of v, $F(\sigma)$ and the lines meeting it in at least 2 points form a projective plane $PG(2, q)$ with q odd, and $C_0(\sigma)^{F(\sigma)}$ is $PSL(3, q)$. In particular, $C_\Pi(\sigma)$ is elementary abelian of order q^2, and $C(\sigma)_{yB}$ acts irreducibly on $C_\Pi(\sigma)$. By the Brauer-Wielandt Theorem [16], $|\Pi|$ divides $|C_\Pi(\sigma)||C_\Pi(\sigma')||C_\Pi(\sigma\sigma')|$. Thus, Π is nilpotent. One of $\sigma, \sigma', \sigma\sigma'$, say σ, centralizes an element $\neq 1$ of $Z(\Pi)$. As $C(\sigma)_{yB}$ acts irreducibly on $C_\Pi(\sigma)$, $C_\Pi(\sigma) \leqq Z(\Pi)$. However, σ, σ' and $\sigma\sigma'$ are conjugate in Γ_{yB}, so that also $C_\Pi(\sigma')$ and $C_\Pi(\sigma\sigma')$ are inside $Z(\Pi)$. Then

$$\Pi = C_\Pi(\sigma)\, C_\Pi(\sigma')\, C_\Pi(\sigma\sigma')$$

is abelian, contradicting Lemma 5(ii).

Thus, no 3 points of $F(\sigma)$ are collinear. We next note that $|F(\sigma) \cap F(\sigma')| = 1$. For otherwise, as $|F(\sigma)|$ is odd and σ' acts on $F(\sigma)$, we can find 3 points y, y_1, y_2 in $F(\sigma) \cap F(\sigma')$. Set $Y = y_1 y_2$. Then $\langle\sigma, \sigma'\rangle \leqq \Gamma_{yy_1y_2}$ and each element $\neq 1$ of $\langle\sigma, \sigma'\rangle$ inverts $\Pi(y\,y_1)_Y$ (see Lemma 1), which is impossible.

By Lemma 2 and the Brauer-Wielandt Theorem [16], $|\Pi| = c^3$ where $c = |C_\Pi(\sigma)|$ is odd. Suppose that σ fixes a line B' not meeting $F(\sigma)$. Then by Lemma 2, $c+2 = |F(\sigma)|$ divides $|\Pi(B')| = c^3$, a contradiction.

Thus, σ fixes just $\frac{1}{2}(c+2)(c+1)$ lines of $\mathscr{D}$. However, if $z \notin F(\sigma)$ then σ fixes $z z^\sigma$. Thus, $\langle\sigma\rangle$ has precisely $(v-(c+2))/2 = \frac{1}{2}(c+2)(c+1)\cdot(k-2)/2$ non-trivial orbits. Also, $(k-1)|(v-k) = c^3$ (Lemma 4(ii)). Then $2(1-(c+2)) \equiv (c+2)(c+1)(-1) \pmod{k-1}$, so that $2(-c-1) \equiv -(c+2)(c+1)$, or $c(c+1) \equiv 0 \equiv c^3 \pmod{k-1}$. Thus, $c \geqq k-1$. However, $k + c^3 - c - 2 = \frac{1}{2}(c+2)(c+1)(k-2)$ implies that $k = 2(c^2+c+2)/(c+3) > c+1$, a contradiction.

Lemma 10. *k is odd.*

Proof ([9]). Suppose that k is even. An involution $\sigma \in \Pi$ moves all points $\notin B$ (Lemma 8), and thus fixes no line $\neq B$ meeting B. σ fixes $B^* = y\,y^\sigma$ and fixes no point of B^*. By Lemma 1 and [3], either Γ_{yB}^B has a normal subgroup $PSL(2, q)$, $q > 3$, acting on B in its usual representation, or Γ_{yB}^B is solvable.

In either case, the conjugates of σ under Γ_{xB^*} generate a group K transitive on B^*, and we can find $\gamma \in \Gamma_{xB^*}$ such that $\langle\sigma, \sigma^\gamma\rangle$ acts on B^* as a Klein group. By Lemmas 8(i) and 5(ii), $|\Gamma(B^*)_x|$ is odd, so that $\langle\sigma, \sigma^\gamma\rangle$ is itself a Klein group. There is an involution $\tau \in \Pi(B^*)$ centralized by $\langle\sigma, \sigma^\gamma\rangle$. Then τ fixes both B and B^γ. Since $\tau \in \Pi(B^*)$ and $x \in B \cap B^\gamma$, we must have $B = B^\gamma$. It follows that $K \leqq \Pi(B)_{B^*}$. However, $1 \neq K^{B^*} \trianglelefteq \Gamma_{xB^*}^{B^*}$ and K^{B^*} is either a sharply transitive group or a Frobenius group (Lemma 8). Consequently, K^{B^*} is sharply transitive and Γ_{yB}^B is solvable. Also, $K = \Pi(B)_{B^*}$ is faithful on B^*. In particular, K is elementary abelian of order k. Interchanging the roles of σ and τ we find that $\Pi(B^*)_B$ is faithful and sharply transitive on B.

Let m be the number of involutions in $\Gamma(B)$. Count in 2 ways the number of ordered triples (x, x', τ) with τ an involution fixing some line pointwise and $x \neq x' = x^\tau$:

$$\frac{v(v-1)}{k(k-1)}\, m(v-k) = v(v-1)\cdot\frac{v-k}{k}\cdot 1.$$

Thus, $m=k-1$ and K contains all involutions of $\Gamma(B)$. In particular, $K \trianglelefteq \Gamma(B)$. It follows that Γ_x has a normal subgroup $\Delta > \Gamma(x) K$ such that $\Delta/\Gamma(x)$ acts on the lines on x as $PSL(2, 2^e)$, $Sz(2^e)$ or $PSU(3, 2^e)$ in its usual 2-transitive representation (Shult [11]).

Consequently, $r-1$ is a power of 2. Since $|\Gamma(x)|$ is odd (Lemma 5(iii)), a Sylow 2-subgroup Σ of $\Gamma(B)$ is faithful and sharply transitive on the lines $\neq B$ on x. In view of the action of $\hat{\Gamma}_x$, we must have $\Sigma \leqq \Delta$. Then $K=Z(\Sigma)$ has order $2^e=k$.

Now $\Gamma(B)=\Pi$. For $|\Gamma(B)_y| \mid k-2$ by Lemma 8(i). Also, $K \trianglelefteq \Gamma_B$ implies that $\Gamma(B)_y$ fixes $y^K=B^*$. Then $\Gamma(B)_y$ fixes $\geqq 2$ points of B^* and hence is trivial by Lemma 8(i).

Let $x \in X \neq B$. Then $[\Gamma(X)_B, \Gamma(B)_X] \leqq \Gamma(X) \cap \Gamma(B)=1$, and $(\Gamma_x)_{BX}$ contains the direct product of two normal subgroups of order 2^e-1 which are conjugate in Γ_x. This is possible only if $\Gamma(x) \neq 1$.

Let $1 \neq \alpha \in \Gamma(x)$, and let $\alpha'=\alpha^{\tau'}$ with τ' an involution in $\Gamma(B^*)_B$. Then $\alpha^{-1}\alpha'=\beta$ acts on B as an involution without fixed points, so that $\beta\tau \in \Gamma(B)$ for some involution $\tau \in \Gamma(B^*)_B$.

By Lemma 5(i), $[\Gamma(B), \alpha]=[\Gamma(B), \alpha']=1$, so that $1=[\Gamma(B), \beta]=[\beta\tau, \beta]=[\tau, \beta]$ and β fixes B^*. Then $\beta\sigma'$ fixes y for some $\sigma' \in K$. Since $\Sigma \leqq \Pi$ centralizes both β and σ', $\beta\sigma'$ fixes at least $|\Sigma|=r-1$ points, including all of B^*. However, $r-1>k$ by Lemma 4, so that $\Delta/\Gamma(x)$ is not $PSL(2, k)$. Then $r-1 \geqq k^2 > k+1$, contradicting Lemma 8.

Lemma 11. (i) *Π is not nilpotent.*

(ii) $\Pi=\Gamma(B)$.

(iii) *Each element of Γ_{yB} of prime order fixes at least 2 points of $S-B$.*

Proof. (i) Suppose that Π is nilpotent, and let σ be an involution in $Z(\Pi)$. By Lemma 5(i), σ fixes no line $\neq B$ on x. Since σ fixes $y y^\sigma$, it follows that k is even, contradicting Lemma 10.

(ii) Lemma 8(ii) and (i).

(iii) Otherwise, such an element would be fixed-point-free on Π, contradicting (i) and a result of Thompson [13].

Lemma 12. *Γ_{yB}^B has no normal subgroup sharply 2-transitive on B.*

Proof ([9]). Otherwise $\Gamma(xy)_B$ has a unique involution. As k is odd, each involution σ in $\Gamma(B)$ fixes $(v-k)/(k-1)=r-1$ lines $\neq B$, each of which meets B. If $\Gamma(B)$ has m involutions then, as in Lemma 10,

$$\frac{v(v-1)}{k(k-1)} m(v-k)=v(v-1)\cdot\frac{v-k}{k-1}\cdot 1,$$

or $m=k$.

$|\Gamma(x)|$ is odd. For, let α be an involution in $\Gamma(x)$. Then $\alpha=\sigma$ on each fixed line $\neq B$ of σ on x, so that σ fixes at most k lines $\neq B$, contradicting Lemma 4(ii).

We may assume that σ fixes $xy=X$. Then $\Gamma(X)_B$ centralizes the unique involution σ of $\Gamma(B)_X$. In particular, $C(\sigma)_x$ is transitive on $B-\{x\}$. As some fixed line of σ is not on x (Lemma 5(i)), $C(\sigma)$ is transitive on B. Thus, σ fixes $1+(r-1)/k$ lines on x.

$\Gamma(B)_X \times \Gamma(X)_B$ has just 3 involutions: σ, τ and $\sigma\tau$, where $\tau \in \Gamma(X)_B$. An involution $\neq \sigma, \tau$ in Γ_{BX} fixing a line pointwise must agree with $\sigma\tau$ on B and X, hence is $\sigma\tau$ (Lemma 11). There are thus $j=2$ or 3 involutions in Γ_{BY} fixing lines pointwise. Count in 2 ways the number of ordered triples (X_1, X_2, ρ) with $X_1, \neq X_2$ lines on x and ρ an involution fixing X_1 and X_2 which fixes some line pointwise:

$$r(r-1)\cdot j = rm \cdot (1+(r-1)/k)\cdot(r-1)/k.$$

Thus, $j-1=(r-1)/k$. By Lemma 4, $j=3$ and $r-1=2k$.

Define a Steiner triple system $\mathscr{S}$ as follows: points are the lines on x; triples consist of 3 distinct lines X_1, X_2, X_3 on x such that the involution in $\Gamma(X_1)_{X_2}$ fixes X_3. Since $1+(r-1)/k=3$, X_1 and X_2 determine X_3 uniquely, and it is easy to see that X_1 and X_3 or X_2 and X_3 determine X_2 or X_1, respectively, in the same manner.

Clearly, $\hat{\Gamma}_x$ is 2-transitive, and $1+(r-1)/k=3$ implies that there is a Klein group fixing a triple elementwise in which no involution fixes more than 3 points of $\mathscr{S}$. By a result of Hall ([2], p. 100) any 3 points of $\mathscr{S}$, not in a triple, generate a subsystem $PG(2,2)$ of $\mathscr{S}$. Thus, $\mathscr{S}$ consists of the points and lines of $PG(d,2)$ for some $d \geqq 2$ (Veblen and Young [14]), and it is easy to see that $d \leqq 3$. Now $\Gamma \approx A_7$, contradicting the minimality of v.

We can now complete the proof of Theorem 1′. Recall that Γ_{yB} is faithful on B.

By Lemmas 1, 10, and 12 and a result of Shult [12], Γ_{yB} has a normal subgroup Δ acting (faithfully) on B as $PSL(2, 2^a)$, $Sz(2^a)$ or $PSU(3, 2^a)$ in its usual 2-transitive representation. Here $2^a > 2$. Let $x' \in B - \{x\}$.

We claim that $|\Gamma_{yxx'}|$ is even. For otherwise, let $\gamma \in \Delta_{xx'}$ have prime order. By Lemma 11 (iii), γ fixes a point $\neq y$ not in xx'. Similarly, γ fixes a point $\neq x'$ not in xy. Thus, γ fixes a quadrangle pointwise. Also, from the structure of Δ we find that γ is inverted by an element of Δ. Then γ is also inverted by some element of $\Gamma_{yxx'}$ (Lemma 7). Since we are assuming that $|\Gamma_{yxx'}|$ is odd, this is impossible.

Since $\Gamma_{yxx'}$ has even order, Δ is not $Sz(2^a)$. Let α be an involution in $\Gamma_{yxx'}$. Then α fixes 2^e+1 points of B, where $e=a/2$ if Δ is $PSL(2, 2^a)$ and $e=a$ if Δ is $PSU(3, 2^a)$. Moreover, $C_\Delta(\alpha)$ is $PSL(2, 2^e)$, and acts on $F(\alpha) \cap B$ in its usual 2-transitive representation.

Since $v-k$ is even, α fixes a point of $B-\{y\}$. Thus, $F(\alpha)$ contains a quadrangle. Also, $|F(\alpha)-F(\alpha)\cap B|$ is even, so that $|F(\alpha)|$ is odd. We can apply Lemma 6(i) to $C_0(\alpha)^{F(\alpha)}$. The points of $F(\alpha)$, together with the lines meeting $F(\alpha)$ at least twice, form one of the following geometries: (i) $PG(2, 2^e)$, (ii) $AG(2,3)$, or (iii) the points and lines of $PG(3,2)$. Moreover, if (ii) or (iii) holds then $2^e+1=3$, so that since $a \geqq 2$ and $e=a$ or $a/2$ we must have $a=2$ and $k=2^2+1=5$.

(i) Here each fixed line X of α meets $F(\alpha)$ in 2^e+1 points. For, $|X - X \cap F(\alpha)|$ is even. If $|X \cap F(\alpha)| \neq 2^e+1$ then $|X \cap F(\alpha)|=1$. By Lemma 2, $C_{\Pi(X)}(\alpha)$ is transitive on $|F(\alpha)-X\cap F(\alpha)|$, and hence contains an involution fixing a single point of $F(\alpha)$, which is impossible.

Thus, $\langle\alpha\rangle$ has

$$(v-(2^{2e}+2^e+1))/2=(2^{2e}+2^e+1)(k-(2^e+1))/2$$

non-trivial orbits. If Δ is $PSL(2, 2^{2e})$ then $k=2^{2e}+1$, so that $v-k=2^{4e}$, whereas Π is not nilpotent. Thus, Δ is $PSU(3, 2^e)$ and $k=2^{3e}+1$. Then

$$r(k-1)<v=(2^{2e}+2^e+1)(k-2^e)<2^{3e}(k-1)<k(k-1),$$

contradicting Lemma 4(ii).

(ii) This time each fixed line of α meets $F(\alpha)$ in 1 or 3 points. Suppose that α fixes j lines meeting $F(\alpha)$ only in x. Then $\langle\alpha\rangle$ has

$$(v-9)/2=12(5-3)/2+9j(5-1)/2$$

non-trivial orbits. Thus, $v\equiv 0 \pmod 3$. However, $|C_\Pi(\alpha)|=9-3$ implies that $v\equiv k=5 \pmod 3$, a contradiction.

(iii) As above, each fixed line of α meets $F(\alpha)$ in 3 points, and $(v-15)/2=35(5-3)/2$. Then $v-k=80$, whereas $|C_\Pi(\alpha)|=12$ divides $v-k$.

This contradiction completes the proofs of Theorems 1′ and 1.

Precisely as Theorem 1 was a consequence of Theorem 1′, Theorem 2 follows from the next result.

Theorem 2′. *Let Γ be an automorphism group of a design $\mathcal{D}$ 2-transitive on points and transitive on blocks. Let B be a block and suppose that Γ_B has a normal nilpotent subgroup $\Pi\leqq\Gamma(B)$ transitive on the set of points not on B. Then one of the following holds:*

(i) *$\mathcal{D}$ consists of the points and hyperplanes of $PG(d, q)$ and $\Gamma\geqq PSL(d+1, q)$;*

(ii) *$\mathcal{D}$ consists of the points and hyperplanes of $AG(d, 2)$ and $\Gamma=ASL(d, 2)$; or*

(iii) *$\mathcal{D}$ is the design associated with M_v, $v=22$, 23 or 24, and Γ is M_{22},* Aut *M_{22}, M_{23} or M_{24}.*

Proof. Let $\mathcal{D}$ and Γ yield a counterexample with v minimal. Once again Γ is not 3-transitive. Our notation is the same as before. Lines have at least 3 points.

By Theorem 1′, $\Pi_y\neq 1$.

Lemma 13. *The points and lines of $\mathcal{D}$ do not form $PG(d, q)$.*

Proof. Otherwise, B is an e-subspace for some e, $1\leqq e<d-1$. Then $v-k=(q^{d+1}-q^{e+1})/(q-1)$ is divisible by p, where p is the prime dividing q. Let α be an element of order p in $Z(\Pi)$. Then α fixes no point not in B and fixes some $(e+1)$-space containing B. Since Π is transitive on the $(e+1)$-spaces containing B, α fixes each such $(e+1)$-space and Π is transitive on the hyperplanes containing B.

On the other hand, as in Lemma 3 we may assume that Π induces a collineation group of $PG(d-e, q)$ in which α is a non-trivial elation. Thus, Π fixes a hyperplane containing B, namely, the axis of α. This is a contradiction.

Lemma 14. *Blocks are lines.*

Proof. This is proved precisely as in Lemma 4.

Lemma 15. *Let $1 \neq \alpha \in Z(\Pi)$. Then α fixes no line meeting B.*

Proof. Suppose that α fixes a line meeting B at x. The transitivity of Π implies that $\alpha \in \Gamma(x)$. Thus, $Z(\Pi) \trianglelefteq \Gamma_B$, $Z(\Pi) \cap \Gamma(x) \neq 1$ and $Z(\Pi)$ acts regularly on $S-B$. In view of Lemma 13, this contradicts [8].

Lemma 16. *Let $y \notin B$ and let $\gamma \in \Pi_y$ have prime order p. If x_1 and y_1 are distinct points of $F(\gamma)$ then the line $x_1 y_1$ joining them is contained in $F(\gamma)$.*

Proof. Since γ normalizes $\Pi(x_1 y_1)$ it centralizes an element $\alpha \in Z(\Pi(x_1 y_1))$ of order p. By Lemma 15, α fixes no line $\neq x_1 y_1$ on x_1. By Gleason's Lemma ([2], p. 191), $C(\gamma)_{x_1}$ is transitive on the lines through x_1 containing at least 2 fixed points of γ.

First, take $x_1 = x$ in order to find that the lemma holds for each line $x y_1$ meeting $F(\gamma)$ at least twice. Then take $x_1 \neq x$, so that $x x_1 \subseteq F(\gamma)$ and hence, by the preceding paragraph, $x_1 y_1 \subseteq F(\gamma)$.

We can now complete the proof of Theorem 2′. Let $\mathscr{D}^*$ be a set of points and lines of $\mathscr{D}$ minimal with respect to the properties: (i) $\mathscr{D}^*$ contains 3 non-collinear points, (ii) if x_1 and y_1 are distinct points of $\mathscr{D}^*$ then $x_1 y_1$ is in $\mathscr{D}^*$, and (iii) if a line is in $\mathscr{D}^*$ so are all of its points. By Lemma 16, $\mathscr{D}^*$ is not all of $\mathscr{D}$.

We may assume that B and y are in $\mathscr{D}^*$. Clearly, $\mathscr{D}^*$ contains a quadrangle.

Let $y \neq y' \notin B$, where $y' \in \mathscr{D}^*$. Then $y' = y^\beta$ for some $\beta \in \Pi$. It follows that $\mathscr{D}^* \cap \mathscr{D}^{*\beta}$ contains B and y'. Then $\mathscr{D}^* \cap \mathscr{D}^{*\beta}$ satisfies (i), (ii), and (iii), so that $\mathscr{D}^{*\beta} = \mathscr{D}^*$.

Thus, $\mathscr{D}^*$ and $(\Gamma_{\mathscr{D}^*})^{\mathscr{D}^*}$ satisfy the hypotheses of Theorem 2′. Consequently, $\mathscr{D}^*$ is a projective plane. Since Γ is transitive on triples of non-collinear points, $\mathscr{D}$ is a projective space (Veblen and Young [14]). This contradicts Lemma 13, and completes the proof of Theorems 2′ and 2.

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