Commutativity in Finite Planes of Type I-4

By

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A projective plane is of Lenz-Barlotti type I-4 if it is (A, BC)-, (B, CA)-, and (C, AB)-transitive for three noncollinear points A, B, C, and is not (P, m)-transitive for any other point-line pair (P, m) (see [3, § 3.1]). The purpose of this paper is the proof of the following result.

Theorem 1. *In a finite projective plane of type* I-4, *the group of* (A, BC)*-homologies is commutative.*

This generalizes a result of Hughes [4, 5], who proved Theorem 1 when the plane has order $\leqq 250$. Our proof essentially amounts to formalizing Hughes' approach, and then applying results on the structure of a finite group having a nontrivial normal partition [1, 2, 7].

Coordinatizing a plane of type I-4 in the usual manner with $U = A$, $V = B$, and $O = C$ (see [3, § 3.1]), we obtain a linear planar ternary ring which has associative multiplication and satisfies both distributive laws. Such a ternary ring is called a *planar division neo-ring* (PDNR). The multiplicative group of a PDNR is isomorphic to the group of all (A, BC)-homologies. Thus, Theorem 1 will follow from the following generalization of Wedderburn's Theorem.

Theorem 2. *Finite PDNR's have commutative multiplication.*

It should be noted that the only known finite PDNR's are finite fields. That is, no examples of finite planes of type I-4 are known to exist. The question of existence has not yet been settled (see [4, 5, 6]). Infinite planes of type I-4 do exist. Some examples can be found in the Appendix of [5].

Proof of Theorem 2. Assume the theorem is not true and let R be a PDNR of smallest order n for which $G = R^*$ is not abelian. Set $Z = Z(G)$ and $\bar{G} = G/Z$. Bars will always denote images in $\bar{G}$.

Let $\mathscr{A}$ be the set of all subgroups of G which are maximal with respect to being abelian. Let $\bar{\mathscr{A}}$ be its image in $\bar{G}$. Clearly $\bar{G} \notin \bar{\mathscr{A}}$ and $Z \notin \mathscr{A}$.

We continue the proof with a series of lemmas.

Lemma 1. *Let X be any subgroup of G which is not contained in Z.*

(a) *$C_G(X)$ is abelian.*

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(b) *If X is abelian, $C_G(X) \in \mathscr{A}$.*

(c) *If X is non-abelian, $C_G(X) = Z$.*

Proof. (a): Consider $S = \{r \in R \mid rx = xr$ for all $x \in X\}$. Since $G = R^*$ is a group and both distributive laws hold, S is closed under multiplication and addition, and hence is a sub-PDNR [5, Theorem I.2]. Note that $S = C_G(X) \cup \{0\}$. Since $X \nleqq Z$, $C_G(X) < G$, and the minimality of $|R|$ implies that $C_G(X)$ is abelian.

(b): If X is abelian, X is contained in $C_G(X)$, so that any member of $\mathscr{A}$ which contains $C_G(X)$ must also centralize X.

(c): Since $X \leqq C_G(C_G(X))$, if X is not abelian then $C_G(C_G(X))$ is not abelian. By (a), this is possible only if $C_G(X) \leqq Z$. Clearly, $C_G(X) \geqq Z$ also holds.

Lemma 2. *Distinct members of $\mathscr{A}$ have intersection Z. If $A \in \mathscr{A}$, then $|Z|^4 < |A|^2 < |G| = n - 1$.*

Proof. Let $A_1, A_2 \in \mathscr{A}$, $A_1 \neq A_2$. Then $C_G(A_1 \cap A_2)$ contains A_1 and A_2 and hence is not in $\mathscr{A}$, so $Z \leqq A_1 \cap A_2 \leqq C_G(C_G(A_1 \cap A_2)) = Z$ by Lemma 1(c). Next, let $A \in \mathscr{A}$. As in the proof of Lemma 1(a), $A \cup \{0\}$ is a proper sub-PDNR of R. By Bruck's lemma on the order of subplanes [3, 3.2.18], we have

$$(|Z| + 1)^4 \leqq (|A| + 1)^2 \leqq |G| + 1$$

since $Z \cup \{0\}$ is a proper sub-PDNR of $A \cup \{0\}$. The lemma now follows easily.

In particular, Lemma 2 implies that $\bar{\mathscr{A}}$ is a *nontrivial normal partition* of $\bar{G}$: each nontrivial element of $\bar{G}$ is in a unique element of $\bar{\mathscr{A}}$, $\bar{G} \notin \bar{\mathscr{A}}$, and conjugates of members of $\bar{\mathscr{A}}$ are in $\bar{\mathscr{A}}$.

Lemma 3. *Z contains a Sylow 2-subgroup of G.*

Proof. Deny this. Then n is odd. Let g be a 2-element satisfying $g \notin Z$ but $g^2 \in Z$. Let α be the inner automorphism determined by g, extended to R by $0^\alpha = 0$. By the distributivity of R, α is an automorphism of R. Since $g^2 \in Z$, $\alpha^2 = 1$, and hence α induces a Baer involution of the plane coordinatized by R (see [3, 4.1.9]). Thus, $|C_G(g)| = m - 1$ where $n = m^2$.

Since $(n-1)/(m-1) = m + 1$ is even, G has a non-abelian Sylow 2-subgroup. Because the multiplicative group of a PDNR has at most one involution [5, Theorem II.3], G contains a quaternion group of order 8, say $\langle g, h\rangle$. Let β be the inner automorphism determined by h, again extended to R. Then α and β commute since $g^2 = h^2 = (gh)^2 \in Z$, so that β induces the identity or an involution on $S = C_G(g) \cup \{0\}$. However, $h \notin C_G(g)$, so $C_G(g) \cap C_G(h) = Z$ (by Lemmas 1(b) and 2) and β induces an involution on S. It follows that $m = s^2$ where $|Z| = s - 1$ and $|C_G(g)| = s^2 - 1$.

Since $4 \nmid s^2 + 1$, there exists an element x in G having odd prime order dividing $(n-1)/(m-1) = s^2 + 1$. As in Lemma 2, $|C_G(x)| \leqq \sqrt{n} - 1 = s^2 - 1$ and $s^2 = (|Z| + 1)^2 \leqq |C_G(x)| + 1$. Consequently, $|C_G(x)| = s^2 - 1$, which is not divisible by the order of x, a contradiction.

Lemma 4. *$\bar{G}$ is solvable.*

Proof. Suzuki ([7, Theorem 1]) has shown that a group of odd order having a nontrivial normal nilpotent partition is solvable.

Lemma 5. *Let P be a non-abelian p-subgroup of G. Then there exists an element y in $P - Z(P)$ such that $C_G(y) \nleqq PZ$.*

Proof. Deny this. Then $C_G(y) \leqq PZ$ for all $y \in P - Z(P)$. Let $X > Z(P)$ be a normal subgroup of P such that $[X : Z(P)] = p$. Then X is abelian and $C_P(X) < P$. Also, $[P : C_P(X)] \leqq |Z(P)|$. To see this, take $g \in X - Z(P)$. Then $C_P(g) = C_P(X)$, and hence $[P : C_P(X)] = [P : C_P(g)]$ is the number of conjugates of g in P. Since $X \lhd P$, these conjugates are in X, so $[P : C_P(X)] < |X| = p|Z(P)|$. Both sides of this inequality are powers of p, so that $[P : C_P(X)] \leqq |Z(P)|$.

Let $y \in P - C_P(X)$. By our assumption, $C_G(y) = C_P(y)Z$. Here $C_P(y) \cap Z = P \cap Z = Z(P)$ by Lemma 1 (c). Consequently, using Lemma 2 we find

$$|Z|^2 < |C_G(y)| = |C_P(y)|\,|Z|/|Z(P)|,$$
$$|Z(P)|^2 \leqq |Z|\,|Z(P)| < |C_P(y)|,$$

and hence

$$|Z(P)| < |C_P(y)|/|Z(P)| = |C_P(X)\,C_P(y)|/|C_P(X)| \leqq \leqq [P : C_P(X)] \leqq |Z(P)|$$

(since by Lemma 1 (b), $C_P(X) \cap C_P(y) = P \cap Z = Z(P)$). This is impossible.

Lemma 6. *$|\bar{G}|$ is not a prime power.*

Proof. If it is, G is nilpotent and hence has a normal, non-abelian Sylow p-subgroup P. By Lemma 1 (c), $C_G(P) = Z$, and since P must centralize all the other Sylow subgroups of G, we have $G = PZ$. This is not possible by Lemma 5.

Lemma 7. *$\bar{\mathscr{A}}$ is not a Frobenius partition.*

Proof. Suppose that $\bar{\mathscr{A}}$ is a Frobenius partition. Then by definition (see [2, p. 333]), some $\bar{A} \in \bar{\mathscr{A}}$ is its own normalizer in $\bar{G}$, and $\bar{G}$ is a Frobenius group. Thus, $\bar{G}$ has a Frobenius kernel $\bar{K}$. By a result of Thompson [8], $\bar{K}$ is nilpotent, and hence so is its preimage K.

We next show that K is abelian. If not, K has a non-abelian Sylow p-subgroup P. By Lemma 1 (c), $C_G(P) = Z$, and hence $K = PZ$ since K is nilpotent and contains Z. Since $\bar{G}$ is Frobenius, for any element $k \in K - Z$ we have that $C_{\bar{G}}(\bar{k}) \leqq \bar{K}$, from which $C_G(k) \leqq K$ follows. Hence, for all $y \in P - Z(P)$, $C_G(y) \leqq K = PZ$. By Lemma 5, however, this is impossible, so K is abelian.

Since $\bar{G} = \bar{K}\bar{A}$ where $\bar{A}$ is as in the first paragraph, we have $G = KA$. Hence, $|G| \leqq |K|\,|A|$. Two applications of Lemma 2 now yield

$$|G|^2 \leqq |K|^2\,|A|^2 < |G|\,|G|,$$

which is a contradiction.

We can now complete the proof of our theorem. $\bar{G}$ is solvable, so its Fitting subgroup is nontrivial. Since $\bar{G}$ has odd order, a theorem of Baer [1, Satz A] implies that

$\bar{\mathscr{A}}$ is not simple (see [2, p. 333] for the definition). Since $\bar{G}$ is not a p-group and $\bar{\mathscr{A}}$ is not a Frobenius partition, another theorem of Baer [2, Satz 5.1] now implies that some $\bar{A} \in \bar{\mathscr{A}}$ is normal and of prime index p in $\bar{G}$, with p dividing $|\bar{A}|$. We now have $G = A\langle x\rangle$ with $|x|$ a power of p. Since A contains Z, by Lemma 2

$$|G| = |A|\,|\bar{x}| = p|A| \leqq |A|^2 < |G|.$$

This contradiction completes the proof.

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