Some Recent Results Concerning Designs and Planes

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This short paper surveys some recent results made possible by the classification of finite simple groups.

1. 2-transitive designs

During the period 1960-1975 there was a great deal of research activity concerning designs $\mathcal{D}$ whose automorphism group $G = \mathrm{Aut}\,\mathcal{D}$ is 2-transitive on points. The results obtained are surveyed in detail in [15]. One of the leaders in this area was Noboru Ito, whose beautiful results [6-11] greatly influenced my research.

During that period it had been hoped that the study of designs might lead to a classification of 2-transitive groups — or, at least, of multiply transitive ones. However, the relatively little work in this direction after 1975 reflected the difficulty of this prolem. Finally, it became apparent that 2-transitive groups would be classified only when all finite simple groups were.

There is now a list of (finite!) 2-transitive groups (e.g., in [16], based on results of Maillet, Curtis-Kantor-

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Seitz, Huppert, Hering, and folklore in the case of the sporadic simple groups). One very easy consequence of this list is the folowing

Theorem 1 ([17]). If $\mathcal{D}$ is a symmetric design with a 2-transitive automorphism group, then $\mathcal{D}$ or its complementary design is a projective space, the 11-point Hadamard design, the 176-point design for the Higman-Sims group, or the 2^{2n}-point design whose (1,-1) incidence matrix is the 2n-fold tensor power of $\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$, $n \geqq 2$.

This theorem and the next one completely settle problems studied at length by Ito and myself:

Theorem 2 ([16]). If $\mathcal{D}$ is a design with $\lambda = 1$ and $k > 2$ whose automorphism group is 2-transitive on points, then $\mathcal{D}$ is one of the following: an affine or projective space; the unital ($v = q^3 + 1$, $k = q+1$) associated with $PSU(3,q)$ or ${}^2G_2(q)$; an affine plane with 3^4 or 3^6 points; or one of two designs having $v = 3^6$ and $k = 3^2$.

The two designs with 3^6 points were discovered by Hering (unpublished). Recent special cases of Theorem 2 are found in [2,4,19]. The proof of Theorem 2 is very easy when G has no regular normal subgroup. In fact, the only delicate part of the proof occurs when G is assumed to be solvable.

There is an analogue of Theorem 2 for t-designs with $\lambda = 1$ having a t-transitive group. In a somewhat similar direction, Theorem 2 immediately produces a classification of all designs with $\lambda = 1$ and $k>2$ whose automorphism group G is transitive on ordered triangles of points. Recently, Li [20] extended this to deal with tansitivity on unordered triangles.

More generally, consider a geometric lattice L. In [16], it was noted that Theorem 2 produces a classification of all such L for which Aut L is transitive on ordered bases. While the corresponding problem for unordered bases remains open, involved arguments in Li [20] settle the cases of dimension 2 or 3.

On the other hand, it is very easy to deduce from Theorem 2 the classification of all t-designs with $\lambda = 1$ whose automorphism group is t-homogeneous on points (see [3] for the case $t = 2$).

Another open problem (somewhat related to Theorem 1) is a classification of all Hadamard matrices whose automorphism group is 2-transitive on rows. This is another important problem studied by Ito [12-14].

2. Projective planes

The first, best and most important result concerning 2-transitive designs was the Ostrom-Wagner Theorem [22], which dealt with the case of projective planes. The proof given 25 years ago is elegant, and led to many other significant results. In particular, Theorem 2 in no way influences the beauty or strength of that fundamental result.

Attempts to generalize the Ostrom-Wagner Theorem began almost immediately after its publication. While there were several directions in which generalizations were studied, the one that concerns us here involves transitivity on flags (incident point-line pairs).

Conjecture A. If π is a finite projective plane such that $G = \mathrm{Aut}\,\pi$ is flag-transitive, then π is desarguesian.

According to a result of Higman-McLaughlin [5], if $\mathcal{D}$ is a design with $\lambda = 1$ and $G \leq \mathrm{Aut}\,\mathcal{D}$ is flag-transitive, then G is primitive on the points of $\mathcal{D}$. Therefore, Conjecture A would follow from

Conjecture B. If π is a finite projective plane such that $G = \mathrm{Aut}\,\pi$ is primitive on points, then π is desarguesian.

These conjectures remain open. The following results handle the group theoretic cases of the conjecture.

Theorem 3 ([18]). Let π be a finite projective plane such that $G = \mathrm{Aut}\,\pi$ is primitive on points (or, less generally, is transitive on flags). If the stabilizer of some flag is not 1, then π is desarguesian.

When the stabilizer of each flag is just 1, π is a difference set plane. If π has order n then the number n^2+n+1 of points is a prime, either $|G| = n^2+n+1$ or G is a Frobenius group, and Conjectures A and B reduce to questions about $\mathbb{Z}_p$. Thus, the open cases of the conjectures are very much not group theoretic.

The proof of Theorem 3 begins with three observations: G is primitive on points, the number n^2+n+1 of points is odd, and G can be assumed to have a simple normal subgroup. All primitive permutation groups of odd degree having a nonsporadic simple normal subgroup were determined in [18]. (Independently, the same result was obtained in [21]. Also independently, the sporadic case was completely handled in [1].) The list of such primitive groups is very long. It contains all maximal parabolic permutation representations of characteristic 2 groups of Lie type, alternating or symmetric groups acting on subsets of a fixed size or on partitions into blocks of equal size, classical groups on orbits of subspaces or on suitable direct sum decompositions of the vector space into subspaces of equal dimension, and so on.

The proof of Theorem 3 then degenerates into a long and tedious case by case elimination of the possibilities in this long list. The main tools involve subplanes. In particular, one can assume that n is a square, $n = m^2$, and that all involutions in G fix exactly m^2+m+1 points. Unfortunately, I was not able to find a uniform approach, and many separate tricks were used.

3. Open problems

The proof of Theorem 3 suggests further problems, such as the following two.

(a) <u>Determine all flag-transitive designs $\mathcal{D}$ with $\lambda = 1$ and v odd but not a prime power.</u> This is related to both sections 1 and 2. By [5], G is primitive on points. One can assume that $G = \mathrm{Aut}\,\mathcal{D}$ has a simple normal subgroup. Then the results in [1,18,21] provide a very long list of possible permutation groups to be checked. Unlike the case in Theorem 3, there is no reason to expect involutions to be well-behaved. Nevertheless, the question seems feasible. Its solution may even produce a better approach to Theorem 3.

(b) <u>Show that if π is finite affine plane and $G = \mathrm{Aut}\,\pi$ is point-primitive then π must be a translation plane.</u> If the plane has order n then there are n^2 points. A standard argument of Ostrom and Wagner [22] settles the case of even n. When n is odd, it is easy to reduce to the situation in which G has a simple normal subgroup. Once again, the results in [1,18,21] produce a long list of primitive groups to check. One can assume that each involution fixes exactly n points. Nevertheless, I have not been able to check the list. The difference between this situation and that of Theorem 3 concerns the action on lines: G was line-transitive in Theorem 3 whereas here G can have many line-orbits of various sizes.

References

1. M. Aschbacher, Overgroups of Sylow subgroups in sporadic groups (to appear).

2. G. Cherlin, L. Harrington and A. Lachlan, $\aleph_0$-categorical $\aleph_0$-stable structure (to appear).

3. A. Delantsheer and J. Doyen (to appear).

4. M.Hall,Jr., Steiner triple systems with a doubly transitive automorphism group (to appear).

5. D. G. Higman and J. E. McLaughlin, Geometric ABA-groups, Ill. J. Math. 5 (1961) 382-397.

6. N. Ito, Uber die Gruppen $PSL_n(q)$, die eine Untergruppe von Primzahlindex erhalten, Acta Sci. Math. (Szeged) 21 (1961) 206-217.

7. N. Ito, On a class of doubly, but not triply transitive permutation groups, Arch. Math. 18 (1967) 564-570.

8. N. Ito, On permutation groups of prime degree p which contain (at least) two classes of conjugate subgroups of index p, Rend. Sem. Mat. Padova 38 (1967) 287- 292.

9. N. Ito, On permutation groups of prime degree p which contain at least two classes of conjugate subgroups of index p, II, Nagoya Math. J. 37 (1970) 201-208.

10. N. Ito, A theorem on Jordan groups, in "Theory of finite groups", Benjamin, New York 1969, pp.47-48.

11. N. Ito, On Wielandt number of transitive permutation groups of prime degree, Acta Math. Acad. Sci. Hungar. 26 (1975) 267-274.

12. N. Ito, Hadamard matrices with "doubly transitive" automorphism groups, Arch. Math. 35 (1980) 100-111.

13. N. Ito and J. S. Leon, An Hadamard matrix of order 36, JCT(A) 34 (1983) 244-247.

14. N. Ito and H. Kimura, Studies on Hadamard matrices with "2-transitive" autmorphism groups, J. Math. Soc. Japan 36(1984) 63-73.

15. W. M. Kantor, 2-transitive designs, in "Proc. Comb. Conf.", Reidel, Dordrecht 1975, pp 44-97.

16. W. M. Kantor, Homogeneous designs and geometric lattices (to appear in JCT(A)).

17. W. M. Kantor, Classification of 2-transitive symmetric designs (to appear in Graphs and Comb.: An Asian Journal).

18. W. M. Kantor, Primitive permutation groups of odd degree, and an application to finite projective planes (submitted).

19. J. D. Key and E. E. Shult, Steiner triple systems with doubly transitive automorphism groups : A corollary to the classificaiton theorem for finite simple group, JCT(A) 36 (1984) 105-110.

20. Hui-Ling Li (to appear).

21. M. Liebeck and J. Saxl, The primitive prmutation groups of odd degree (to appear).

22. T. G. Ostrom and A. Wagner, On projective and affine planes with transitive collineation groups, Math. Z. 71 (1959) 186-199.