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ELATIONS OF DESIGNS

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An elation of a design $\mathscr{D}$ is an automorphism γ of $\mathscr{D}$ fixing some block X pointwise and some point x on X blockwise. Lüneburg [**4**] and I [**2**] have proved results which state that a design admitting many elations and having additional properties must be the design of points and hyperplanes of a finite desarguesian projective space. In this note, additional results of this type will be proved and applied to yield a generalization of a previous result on Jordan groups [**3**]. The proofs were suggested by a result of Hering on elations of finite projective planes [**1**, pp. 122, 190].

Much of our notation can be found in [**1**]. Designs will always satisfy $v \geqq k + 2$, and the blocks will be distinguishable as sets of points. Isomorphic designs will be identified. The complement of the block X is $\mathscr{C}X$. If Γ is an automorphism group of a design, and $x \in X$, then $\Gamma(X)$ and $\Gamma(x)$ are the largest subgroups of Γ fixing X pointwise and x blockwise, respectively. If $\Pi(X) \leqq \Gamma(X)$, then $\Pi(x, X) = \Gamma(x) \cap \Pi(X)$. If $\Pi(X) \leqq \Gamma(X)$ for all X, then, for each block X and each point x, $\Pi(X)^*$ is the set $\bigcup_{y \in X} \Pi(y, X)$ and $\Pi(x)^* = \bigcup_{x \in Y} \Pi(x, Y)$. $[\alpha, \beta]$ is the commutator $\alpha^{-1}\beta^{-1}\alpha\beta$. If g is a power of a prime p and n is an integer, $g \| n$ means that $g | n$ but $pg \nmid n$. A permutation group is said to act regularly if only the identity fixes a point.

LEMMA 1. *Let $\Delta_0, \Delta_1, \ldots, \Delta_s$ be non-trivial normal subgroups of a finite group Δ such that $s \geqq 1$, $\Delta_i \cap \Delta_j = 1$ if $i \neq j$, and*

$$\left(\bigcup_{0 \leqq i \leqq s} \Delta_i\right) \Delta_0 \subseteq \bigcup_{0 \leqq i \leqq s} \Delta_i.$$

Then there is a prime p such that all Δ_i are p-groups.

Proof. Let $\delta_0 \in \Delta_0$ have prime order p. If $\delta \in \Delta_j$, $j > 0$, then

$$[\delta_0, \delta] \in \Delta_0 \cap \Delta_j = 1.$$

Also,

$$\delta\delta_0 \in \bigcup_{\substack{0 \leqq i \leqq s;\\ i \neq j}} \Delta_i.$$

Consequently,

$$(\delta\delta_0)^p = \delta^p \in \Delta_j \cap \left(\bigcup_{\substack{0 \leqq i \leqq s;\\ i \neq j}} \Delta_i\right) = 1.$$

Thus, each Δ_j with $j > 0$ has exponent p. As this determines p uniquely, Δ_0 is also a p-group.

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LEMMA 2. *Let $\mathscr{D}$ be a design, Γ an automorphism group of $\mathscr{D}$, p, q points, and B, C blocks such that $p \in B - B \cap C$, $q \in B \cap C$. Also let $\theta, \theta' \in \Gamma(p, B)$ and $\varphi, \varphi' \in \Gamma(q, C)$. Then*

(i) $[\theta, \varphi] \in \Gamma(q, B)$;

(ii) *If* $[\theta, \varphi] = [\theta', \varphi]$, *then either* $\varphi \in \Gamma(q, C) \cap \Gamma(D)$ *where* $D \neq C$, *or* $\theta'\theta^{-1} \in \Gamma(p, B) \cap \Gamma(C)$; *and*

(iii) *If* $[\theta, \varphi] = [\theta, \varphi']$, *then either* $\varphi'\varphi^{-1} \in \Gamma(q, C) \cap \Gamma(D)$, *where* $D \neq C$, *or* $\theta \in \Gamma(p, B) \cap \Gamma(C)$.

Proof. (i) $\theta^{-1}\varphi^{-1}\theta \in \Gamma(q, C^{\theta})$ and $\varphi^{-1}\theta\varphi \in \Gamma(p^{\varphi}, B)$ imply that

$$[\theta, \varphi] \in \Gamma(q) \cap \Gamma(B) = \Gamma(q, B).$$

(ii) As $\theta'\theta^{-1}$ and φ commute, $\varphi \in \Gamma(q, C) \cap \Gamma(q, C^{\theta'\theta^{-1}})$. If $\theta'\theta^{-1}$ is in $\Gamma(p, B)_C$, it fixes all lines [**1**, p. 65] on p meeting C and consequently is contained in $\Gamma(p, B) \cap \Gamma(C)$.

(iii) As θ and $\varphi^{-1}\varphi'$ commute, $\varphi^{-1}\varphi' \in \Gamma(q, C) \cap \Gamma(q, C^{\theta})$. If $\theta \in \Gamma(p, B)_C$, then $\theta \in \Gamma(p, B) \cap \Gamma(C)$.

THEOREM 1. *Let $\mathscr{D}$ be a design admitting an automorphism group Γ such that, for each block X, Γ_X has a normal subgroup $\Pi(X) \leqq \Gamma(X)$ satisfying the following conditions:*

(i) $\Pi(X^{\gamma}) = \Pi(X)^{\gamma}$ *for all X and all $\gamma \in \Gamma$;*

(ii) $\Pi(x, X) \neq 1$ *whenever $x \in X$; and*

(iii) $\Pi(x, X) \cap \Pi(Y) = 1$ *whenever $x \in X \neq Y$.*

Then $\mathscr{D}$ is the design of points and hyperplanes of a finite projective space, and Γ contains the little projective group.

We remark that the case $\Pi(X) = \Gamma(X)$ of this theorem is only very slightly weaker than the theorem itself, and suffices for our application to Jordan groups. In later results, only the case $\Pi(X) = \Gamma(X)$ will be considered.

Proof. Let X and Y be distinct blocks, and suppose that $x \in X - X \cap Y$ and $y \in X \cap Y$. If $1 \neq \alpha \in \Pi(x, X)$, then as in Lemma 2, $\beta \to [\alpha, \beta]$, $\beta \in \Pi(y, Y)$, defines an injection $\Pi(y, Y) \to \Pi(y, X)$. If $1 \neq \beta \in \Pi(y, Y)$, then $\alpha \to [\alpha, \beta]$, $\alpha \in \Pi(x, X)$, defines an injection $\Pi(x, X) \to \Pi(y, X)$. Then $|\Pi(y, Y)| \leqq |\Pi(y, X)|$ and $|\Pi(x, X)| \leqq |\Pi(y, X)|$. As x and y are any points of X, while X and Y are any blocks on y, it follows that $|\Pi(x, X)| = g$ is independent of the block X and the point $x \in X$. The above mappings are thus bijective.

Let $1 \neq \alpha \in \Pi(x, X)$ and $\gamma \in \Pi(y, X)$. Then $\gamma = [\alpha, \beta]$ for some $\beta \in \Pi(y, Y)$, and $\alpha\gamma \in \Pi(x^{\beta}, X)$. Thus, $\Pi(X)^*$ is a subgroup of $\Pi(X)$. Similarly, $\Pi(x)^*$ is a subgroup of $\Gamma(x)$. By Lemma 1, there is a prime p such that g, $|\Pi(X)^*| = 1 + (g-1)k$, and $|\Pi(x)^*| = 1 + (g-1)r$ are powers of p. In particular, $g \| (k-1)$ and $p \nmid r$. (iii) implies that $\Pi(x)^*$ acts regularly on the blocks not on x. Thus

$$[1 + (g-1)r] \| (b - r) = (v - k)(r/k),$$

so that $[1 + (g-1)r] \| (v-k) \cdot \lambda = (r-\lambda)(k-1)$ since $p \nmid r$. Since $g \| (k-1)$, it follows that

$$[1 + (g-1)r] \| (r-\lambda)g < 2[1 + (g-1)r].$$

Thus, $r = g\lambda + 1$.

If $y \neq x$, then, since $\Pi(x)^*_y$ acts regularly on the blocks not on x,

$$r - \lambda = 1 + (g-1)\lambda = \left| \bigcup_{x,y \in X} \Pi(x, X) \right| \leqq |\Pi(x)^*_y| (r - \lambda).$$

It follows that $\Pi(x)^*$ is transitive on the blocks not on x and $\Pi(x, X)$ acts regularly on $\mathscr{C}X$ when $x \in X$. Then $1 + (g-1)r = |\Pi(x)^*| = b - r$ and each line has at least $g + 1$ points. However, each line has at most $(b-\lambda)/(r-\lambda) = g + 1$ points, and all lines have this many points if and only if $\mathscr{D}$ consists of the points and hyperplanes of a projective space [**1**, pp. 65, 67]. Together with the transitivity of $\Pi(x)^*$, this proves that $\mathscr{D}$ is desarguesian [**1**, p. 126] and Γ contains the little projective group.

COROLLARY 1. *Let $\mathscr{D}$ be a design admitting a 2-transitive automorphism group Γ such that, for each block X, Γ_X has a normal abelian subgroup fixing X pointwise and transitive on $\mathscr{C}X$. Then $\mathscr{D}$ is either the design of points and hyperplanes of a finite desarguesian projective space or of an affine space over* GF(2), *or $v = 22$, 23 or 24 and $\mathscr{D}$ is the design associated with the Mathieu group M_v* (see [**3**]).

Proof. By [**3**, Theorem 6.5], we may assume that lines have more than two points. By [**3**, Lemma 8.1 (ii)], for each $x \in X$ the given subgroup $\Pi(X)$ of $\Gamma(X)$ has a non-trivial element fixing x blockwise. Since $\Pi(X)$ is abelian, it is regular on $\mathscr{C}X$. The result now follows from Theorem 1.

COROLLARY 2. *Let Γ be a 2-transitive but not k-transitive group of finite degree $v \geqq k + 2 > 4$ such that, for some set X of k points, Γ_X has a normal abelian subgroup fixing X pointwise and transitive on the remaining points. Then Γ is similar to one of the following groups in its usual representation: a subgroup of* $\mathrm{P\Gamma L}(d, q)$ *containing* $\mathrm{PSL}(d, q)$ *for some d, q; the full collineation group of* $\mathrm{AG}(d, 2)$ *for some d; the Mathieu group M_v, $v = 22$, 23 or 24; or* $\mathrm{Aut}(M_{22})$.

Proof. Corollary 1 and [**3**, Lemma 3.2 and Theorem 5.3].

COROLLARY 3. *Let $\mathscr{D}$ be a design with $\lambda = 1$ admitting an automorphism group Γ such that*

(i) *For each point x there is a block X on x for which $\Gamma(x, X) \neq 1$;*

(ii) *For each block X there is a point $x \in X$ for which $\Gamma(x, X) \neq 1$; and*

(iii) $\Gamma(x, X) \cap \Gamma(Y) = 1$ *if $x \in X \neq Y$.*

Then $\mathscr{D}$ is a desarguesian plane and Γ contains the little projective group.

Proof. Suppose that $y \in X$. Let $y \neq x \in X$ and $y \in Y \neq X$. If $\Gamma(y, Y) \neq 1 \neq \Gamma(x, X)$, then $\Gamma(y, X) \neq 1$ by Lemma 2. Theorem 1 thus applies.

THEOREM 2. *Let $\mathscr{D}$ be a design with $\lambda = 1$ admitting an automorphism group Γ such that conditions* (ii) *and* (iii) *of Corollary* 3 *hold. Then $\mathscr{D}$ is a projective plane.*

Proof. We assume that $\mathscr{D}$ is not a projective plane, and adopt the following terminology. Lines are blocks. A centre is a point c such that $\Gamma(c, L) \neq 1$ for some line L on c; any other point is a non-centre. A 1-line is a line L such that $\Gamma(c, L) \neq 1$ for exactly one $c \in L$; any other line is called a 2-line.

Let c and d be distinct centres and $L = cd$ the line joining them. Let $\Gamma(x, L) \neq 1, x \in L$, where we may assume that $x \neq c$. Suppose that $\Gamma(c, L') \neq 1$ with $c \in L' \neq L$. By Lemma 2, $\Gamma(c, L) \neq 1$. Thus, the join of two centres is a 2-line. Therefore, 1-lines contain only one centre, and if a line M contains a centre c, then $\Gamma(c, M) \neq 1$.

Let c, d, and L be as above. There is a centre not on L, since otherwise all lines would meet L. Thus, there is a 2-line $L' \neq L$ on c. Let $1 \neq \alpha \in \Gamma(d, L)$ and $\beta \in \Gamma(c, L)$. By Lemma 2, $\gamma \to [\alpha, \gamma]$, $\gamma \in \Gamma(c, L')$, defines an injection $\Gamma(c, L') \to \Gamma(c, L)$. By symmetry, this is a bijection. Then $\beta = [\alpha, \gamma]$ for some $\gamma \in \Gamma(c, L')$, so that $\alpha\beta \in \Gamma(d^\gamma, L)$. It follows from Lemma 1 that $\Gamma(L)^*$ is a p-group for some prime p. As $|\Gamma(c, L')| = |\Gamma(c, L)|$ and the join of two centres is a 2-line, p is the same for all 2-lines.

Let M be a 1-line on c. We know that $\Gamma(c, M) \neq 1$. Let $1 \neq \alpha \in \Gamma(d, L)$. $\delta \to [\alpha, \delta]$, $\delta \in \Gamma(c, M)$, defines an anti-monomorphism $\Gamma(c, M) \to \Gamma(c, L)$. For, if $\delta, \epsilon \in \Gamma(c, M)$, then

$$[\alpha, \delta\epsilon] = [\alpha, \epsilon][\alpha, \delta]^\epsilon = [\alpha, \epsilon][\alpha, \delta]$$

since $[\alpha, \delta] \in \Gamma(c, L)$, $\epsilon \in \Gamma(c, M)$, and $\Gamma(c, L) \cap \Gamma(c, M) = 1$ by (ii). Thus, all elations in Γ are p-elements. By Gleason's Lemma [**1**, p. 191], it follows that for each centre c and non-centre x, Γ_c is transitive on the 2-lines on c, while Γ_x is transitive on the lines on x.

There exist 1-lines. Otherwise, Γ is line-transitive and thus point-transitive [**1**, p. 78]. Then all points are centres, and Corollary 3 yields a contradiction.

Let M be a 1-line and c the centre on M. Γ transitively permutes the lines containing a point $\neq c$ on M, so that all such lines are 1-lines. If N is any line containing a non-centre x, Γ_x has an element mapping N to a line meeting M at a point $\neq c$, and N is a 1-line. Thus, for each 2-line L, $\Gamma(c, L) \neq 1$ for all $c \in L$.

Let $\mathscr{D}^*$ consist of the centres and 2-lines. Then $\mathscr{D}^*$ is a subdesign of $\mathscr{D}$ fixed by Γ. By Corollary 3, $\mathscr{D}^*$ is a projective plane. Let M be a 1-line on a centre c. Then $\Gamma(c, M)$ induces a collineation group of $\mathscr{D}^*$ with centre c. A non-trivial element of $\Gamma(c, M)$ must fix pointwise some line of $\mathscr{D}^*$. This contradicts (iii).

For further results on the planes characterized in Theorem 2, see [**1**, p. 193].

THEOREM 3. *Let $\mathscr{D}$ be a design with $\lambda = 1$ admitting an automorphism group Γ such that*

(i) *For each point x there are at least two blocks X on x for which $\Gamma(x, X) \neq 1$; and*

(ii) $\Gamma(x, X) \cap \Gamma(Y) = 1$ *if* $x \in X \neq Y$.

Then $\mathscr{D}$ is a desarguesian projective plane and Γ contains the little projective group.

Proof. Blocks will again be called lines. A line L is an axis if $\Gamma(c, L) \neq 1$ for some $c \in L$, and a non-axis otherwise. In view of Corollary 3, we may assume that non-axes exist.

As in the proof of Corollary 3, if L is an axis, then $\Gamma(x, L) \neq 1$ for all $x \in L$. Let $c, d \in L$, $c \neq d$. Let M be an axis $\neq L$ on c, and $1 \neq \gamma \in \Gamma(c, M)$. By Lemma 2, $\alpha \to [\alpha, \gamma]$, $\alpha \in \Gamma(d, L)$, defines an injection $\Gamma(d, L) \to \Gamma(c, L)$. By symmetry, $|\Gamma(c, L)| = g(L)$ depends only on L. Similarly, $|\Gamma(c, M)| = g(L)$. By our previous argument there is a prime p such that $\Gamma(L)^*$ and $\Gamma(c)^*$ are p-groups.

Set $g = g(L)$. Then $|\Gamma(L)^*| = 1 + (g - 1)k$ shows that $g \| (k - 1)$. If there are s axes on c, then $|\Gamma(c)^*| = 1 + (g - 1)s$. Suppose that $s < r$. Since $\Gamma(c)^*$ acts regularly on the points $\neq c$ of a non-axisth rough c (by (ii)), $[1 + (g - 1)s] \| (k - 1)$, contradicting $g \| (k - 1)$. Since c is any point, and all lines on c are axes, there are no non-axes, a contradiction.

THEOREM 4. *Let $\mathscr{D}$ be a design with $\lambda > 1$ admitting an automorphism group Γ fixing a block B and satisfying the following conditions:*

(i) *$\Gamma(x, X)$ is non-trivial and acts regularly on $\mathscr{C}X$ whenever $x \in B$ and $x \in X$; and*

(ii) *If X and Y are blocks $\neq B$ such that $B \cap X \cap Y \neq \emptyset$ but $B \cap X \neq B \cap Y$, then $B \cap X \not\supset B \cap Y$.*

Then $\mathscr{D}$ is the design of points and hyperplanes of a projective space.

Proof. Let x and y be distinct points of B, and X a block on x not on y. If $1 \neq \gamma \in \Gamma(x, X)$ then, by Lemma 2, $\beta \to [\gamma, \beta]$, $\beta \in \Gamma(y, B)$, defines an injection $\Gamma(y, B) \to \Gamma(x, B)$. By symmetry, this is bijective and $|\Gamma(x, B)| = g$ is independent of $x \in B$. If $\alpha \in \Gamma(x, B)$, then $\alpha = [\gamma, \beta]$ for some $\beta \in \Gamma(y, B)$, so that $\gamma\alpha \in \Gamma(x)^*$. Thus, $\Gamma(x)^*\Gamma(x, B) \subseteq \Gamma(x)^*$. By Lemma 1, there is a prime p such that $\Gamma(x, B)$ and $\Gamma(x, X)$ are p-groups. Then g is a power of p, and p is independent of the choice of x and X.

Let L be a line contained in B. If $x \in L$, let $x \in X$, $L \not\subset X$. Then a non-trivial element of $\Gamma(x, X)$ is a p-element fixing L but moving all points of $L - \{x\}$. By Gleason's Lemma [**1**, p. 191], Γ is transitive on B.

Let X and Y be distinct blocks $\neq B$ on x, where once again $x \in B$. If there is a point $y \in B \cap Y - B \cap X \cap Y$, let $1 \neq \gamma \in \Gamma(y, Y)$. By Lemma 2, $\alpha \to [\alpha, \gamma]$, $\alpha \in \Gamma(x, X)$, defines an injection $\Gamma(x, X) \to \Gamma(x, Y)$. By (ii) we may use symmetry to deduce that $|\Gamma(x, X)| = |\Gamma(x, Y)|$. Suppose next that $B \cap X = B \cap Y$. If $B \cap X = \{x\}$, choose a block $Z \neq B$ on x meeting B in a point $\neq x$; then $B \cap Z$ properly contains $B \cap X$, contradicting (ii). We can thus find a block Z on x not containing $B \cap X$. Then

$$|\Gamma(x, X)| = |\Gamma(x, Z)| = |\Gamma(x, Y)|.$$

As Γ is transitive on B, $|\Gamma(x, X)| = g'$ is independent of $x \in B$ and $X \neq B$ on x. As already noted, g' is a power of p.

We now prove that $\Gamma(x)^*$ is a group. We have already shown that $\Gamma(x)^*\Gamma(x, B) \subseteq \Gamma(x)^*$. Once again assume that X and Y are distinct blocks $\neq B$ on x such that there is a point $y \in B \cap Y - B \cap X \cap Y$. Let $1 \neq \alpha \in \Gamma(x, X)$ and $\beta \in \Gamma(x, Y)$. By Lemma 2, $\gamma \to [\alpha, \gamma]$, $\gamma \in \Gamma(y, Y)$, defines a bijection $\Gamma(y, Y) \to \Gamma(x, Y)$ so that $\beta = [\alpha, \gamma]$ for some such γ, and $\alpha\beta \in \Gamma(x)^*$.

Now let X and Y be distinct and on x, let $1 \neq \alpha \in \Gamma(x, X)$, $1 \neq \beta \in \Gamma(x, Y)$ and $\alpha\beta \notin \Gamma(x)^*$. Then $B \cap X = B \cap Y$. Let $z \in B - B \cap X$ and $x, z \in Z \neq B$. Also let $1 \neq \gamma \in \Gamma(x, Z)$. As $\delta \to [\alpha, \delta]$, $\delta \in \Gamma(z, Z)$, defines a bijection $\Gamma(z, Z) \to \Gamma(x, Z)$, $\gamma = [\alpha, \delta]$ where $1 \neq \delta \in \Gamma(z, Z)$. Similarly, $\gamma = [\beta^{-1}, \epsilon]$ where $\epsilon \in \Gamma(z, Z)$. Here $\alpha\beta = \delta^{-1}\alpha\delta \cdot \epsilon^{-1}\beta\epsilon \notin \Gamma(x)^*$. Since $\delta^{-1}\alpha\delta \in \Gamma(x, X^{\delta})$ and $\epsilon^{-1}\beta\epsilon \in \Gamma(x, Y^{\epsilon})$, it follows that $B \cap X^{\delta} = B \cap Y^{\epsilon}$. Then $\delta\epsilon^{-1} \in \Gamma(z, Z)$ fixes $B \cap X = B \cap Y$, thus by (ii) fixes a point of $B \cap X - B \cap X \cap Z$, and so is equal to 1 by (i). Then $[\alpha, \delta] = \gamma = [\beta^{-1}, \delta]$, so that $\alpha\beta$ commutes with δ and thus fixes z. As z was arbitrary and $\alpha\beta$ fixes $B \cap X$ pointwise, $\alpha\beta \in \Gamma(x) \cap \Gamma(B) \subseteq \Gamma(x)^*$, a contradiction. This proves that $\Gamma(x)^*$ is a p-group.

If $x \neq y \in B$ and $z \notin B$, then, by (i),

$$|\Gamma(x)^*_y| = 1 + (g - 1) + (g' - 1)(\lambda - 1)$$

and $|\Gamma(x)^*_z| = 1 + (g' - 1)\lambda$ are powers of p. Since

$$1 + (g - 1) + (g' - 1)(\lambda - 1) = [1 + (g' - 1)\lambda] + (g - g'),$$

it follows that $g = g'$. It is now easy to show that $\Gamma(B)^*$ is a group. As in the proof of Theorem 1, $b = gr + 1$ and $r = g\lambda + 1$. $bk = vr$ and $\lambda(v - 1) = r(k - 1)$ imply that $\mathscr{D}$ is symmetric, so that $|\Gamma(B)^*| = 1 + (g - 1)k = v - k$ and $\Gamma(B)^*$ is transitive on $\mathscr{C}B$. The theorem now follows from [**1**, p. 85] or [**2**].

COROLLARY 4. *Let $\mathscr{D}$ be a symmetric design with $\lambda > 1$ admitting an automorphism group Γ fixing a block B and such that $\Gamma(x, X) \neq 1$ whenever $x \in B, X$. Then $\mathscr{D}$ is the design of points and hyperplanes of a projective space.*

Proof. [**4**, *Hilfsatz* 10] and Theorem 4.

This corollary is [**4**, *Satz* 10] but without assumption (1) (also see [**1**, p. 86]).

THEOREM 5. *Let $\mathscr{D}$ be a design with $\lambda > 1$ admitting an automorphism group Γ fixing a point q and such that:*

(i) $\Gamma(x, X)$ *is non-trivial and acts regularly on* $\mathscr{C}X$ *whenever* $q, x \in X$;

(ii) *There are no blocks* X *and* Y *such that* $X \cap Y = \{q\}$; *and*

(iii) *A non-trivial element of* $\Gamma(q)$ *fixes pointwise at most* $\lambda + 1$ *blocks not on* q. *Then* $\mathscr{D}$ *is the design of points and hyperplanes of a projective space.*

Proof. Let $q \in X \cap Y, X \neq Y, x \in X - X \cap Y$ and $1 \neq \gamma \in \Gamma(x, X)$. By Lemma 2, $\beta \rightarrow [\gamma, \beta]$, $\beta \in \Gamma(q, Y)$, defines an injection $\Gamma(q, Y) \rightarrow \Gamma(q, X)$. Then $|\Gamma(q, Y)| \leqq |\Gamma(q, X)|$ implies that $|\Gamma(q, X)| = g$ is independent of the block X on q. If $\alpha \in \Gamma(q, X)$, then $\alpha = [\gamma, \beta]$ with $\beta \in \Gamma(q, Y)$, and $\gamma\alpha \in \Gamma(x^\beta, X)$. Thus, $\Gamma(X)^* \Gamma(q, X) \subseteq \Gamma(X)^*$.

Let q, x, and y be distinct points of X. If there is a block Y on q and y but not on x, let $1 \neq \gamma \in \Gamma(y, Y)$. By Lemma 2, $\alpha \rightarrow [\alpha, \gamma]$, $\alpha \in \Gamma(x, X)$, defines an injection $\Gamma(x, X) \rightarrow \Gamma(y, X)$. Thus, $|\Gamma(x, X)| \leqq |\Gamma(y, X)|$, so that $|\Gamma(x, X)| = |\Gamma(y, X)|$. If, however, q, x, and y are collinear, let $z \in X - qx$. Then $|\Gamma(x, X)| = |\Gamma(z, X)| = |\Gamma(y, X)| = g(X)$ is independent of $x \in X$, $x \neq q$.

Let X and X' be distinct blocks on q, so that $|X \cap X'| \geqq 2$ by (ii). Let $q \neq x \in X \cap X'$, $z \in X' - X \cap X'$, and $1 \neq \delta \in \Gamma(z, X')$. By Lemma 2, $\alpha \rightarrow [\alpha, \delta]$, $\alpha \in \Gamma(x, X)$, defines an injection $\Gamma(x, X) \rightarrow \Gamma(x, X')$. It follows that $g(X) = g(X') = g'$ is independent of the block X on q.

To show that $\Gamma(X)^*$ is a group when $q \in X$, let q, x, and y be non-collinear points of X, let $q, y \in Y$ and $x \notin Y$. Also let $1 \neq \alpha \in \Gamma(x, X)$ and $\beta \in \Gamma(y, X)$. As usual, $\beta = [\alpha, \gamma]$ for some $\gamma \in \Gamma(y, Y)$. Thus, $\alpha\beta \in \Gamma(X)^*$.

Now let q, x, and y be distinct and on X, let $1 \neq \alpha \in \Gamma(x, X)$, $1 \neq \beta \in \Gamma(y, X)$ and $\alpha\beta \notin \Gamma(X)^*$. Then $qx = qy$. If $q \in Z$ and $x \notin Z$, then by (ii) there is a point $z \neq q$ on $X \cap Z$. Let $1 \neq \gamma \in \Gamma(z, Z)$. As $\delta \rightarrow [\alpha, \delta]$, $\delta \in \Gamma(z, Z)$, defines a bijection $\Gamma(z, Z) \rightarrow \Gamma(z, Z)$, $\gamma = [\alpha, \delta]$ for some such δ, and $\delta \neq 1$. Similarly, $\gamma = [\beta^{-1}, \epsilon]$ for some $\epsilon \in \Gamma(z, Z)$. Since

$$\alpha\beta = \delta^{-1}\alpha\delta \cdot \epsilon^{-1}\beta\epsilon \notin \Gamma(X)^*,$$
$$\delta^{-1}\alpha\delta \in \Gamma(x^\delta, X) \quad \text{and} \quad \epsilon^{-1}\beta\epsilon \in \Gamma(y^\epsilon, X),$$

it follows that $qx^\delta = qy^\epsilon$. Then $\delta\epsilon^{-1} \in \Gamma(z, Z)$ fixes qx and thus $=1$. Then $[\alpha, \delta] = [\beta^{-1}, \delta]$, so that $\alpha\beta$ commutes with δ and thus fixes Z. Since $\alpha\beta$ also fixes all blocks on q and $x, \alpha\beta \in \Gamma(X) \cap \Gamma(q) \subseteq \Gamma(X)^*$, a contradiction. $\Gamma(X)^*$ is thus a group.

By a standard argument, $\Gamma(X)^*$ is an elementary abelian p-group for some prime p. Thus, g, g', and $1 + (g - 1) + (g' - 1)(k - 1)$ are powers of p, and $g || (k - 1)$. By (i) it follows that $g | (v - k)$ but $g \nmid\nmid (v - k)$. It follows that $g || (v - 1)$.

Let $x \neq q$, and let X and Y be distinct blocks on q and x. In the usual way we can define a bijection $\Gamma(x, X) \rightarrow \Gamma(x, Y)$ in order to show that $\Gamma(x)^*$ is a p-group of order $1 + (g' - 1)\lambda$. Then $g' | (\lambda - 1)$. Since $\Gamma(x)^*$ acts regularly on the blocks on q but not x (by (i)), $p | (r - \lambda)$. Thus, $p | (r - 1)$.

Γ is transitive on the points $\neq q$. For, if L is a line not on q and $x \in L$, then

Γ has a p-element fixing x and L but moving all points of $L - \{x\}$. The assertion then follows from Gleason's Lemma [**1**, p. 191].

Since $\Gamma(q, X) \cap \Gamma(q, Y) = 1$ if $q \in X \cap Y$, $X \neq Y$, the subgroup $\overline{\Gamma(q)}$ of Γ generated by $\Gamma(q)^*$ is an elementary abelian p-group. Clearly $\overline{\Gamma(q)} \trianglelefteq \Gamma$. Then all orbits of $\overline{\Gamma(q)}$ of points $\neq q$ have the same length $\bar{g}$. If $q \in X$, then

$$g = |\Gamma(q, X)| \leqq \bar{g}|(v - 1).$$

However, $g||(v - 1)$ and $\bar{g}$ is a power of p. Thus, $\bar{g} = g$.

We now show that $\overline{\Gamma(q)}$ acts regularly on the blocks not on q. For let $\varphi \in \overline{\Gamma(q)}$ fix Z, where $q \notin Z$. φ fixes each block in $Z^{\overline{\Gamma(q)}}$ pointwise, and thus fixes some block $Z' \neq Z$ not on q. Then φ fixes a point $x \neq q$ not on Z. If $\gamma \in \Gamma(x)^*$, then $[\varphi, \gamma] \in \Gamma(x)^* \cap \Gamma(q) = 1$. Thus, φ fixes Z' and all blocks in $Z^{\Gamma(x)^*}$, a total of at least $1 + 1 + (g' - 1)\lambda \geqq \lambda + 2$ blocks. By (iii), $\varphi = 1$, as claimed.

Thus, $|\overline{\Gamma(q)}||(b - r) = (r/k)(v - k) = (r/k\lambda)(k - 1)g^{-1} \cdot g(r - \lambda)$. Since $p \nmid r$ and $g||(k - 1)$, it follows that

$$1 + (g - 1)r \leqq |\overline{\Gamma(q)}||g(r - \lambda) < 2[1 + (g - 1)r].$$

This implies that $|\overline{\Gamma(q)}| = g(r - \lambda)$. If $x \neq q$, then $|\overline{\Gamma(q)}_x| = g(r - \lambda)/\bar{g} = r - \lambda$. Then $\overline{\Gamma(q)}_x$ is transitive on the blocks on x not on q, so that $\overline{\Gamma(q)}$ is transitive on the blocks not on q. Then $b - r = g(r - \lambda)$, or $v - 1 = gk$, and $(v, k) = 1$. Since $b - r = (r/k)(v - k)$ is a power of p and $p \nmid r$, it follows that $r/k = 1$. Corollary 4 now completes the proof.

References

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