Journal of Combinatorial Theory, Series A **98**, 410–415 (2002)
doi:10.1006/jcta.2001.3250, available online at http://www.idealibrary.com on IDEAL®

NOTE

Symmetric Designs from the $G_2(q)$ Generalized Hexagons

U. Dempwolff

Mathematisches Institut, Universität Kaiserslautern, 67653 Kaiserslautern, Germany
E-mail: dempwolff@mathematik.uni-kl.de

and

W. M. Kantor[1]

Department of Mathematics, University of Oregon, Eugene, Oregon 97403
E-mail: kantor@math.uoregon.edu

Received August 9, 2000

We describe symmetric designs $\mathbf{D}$ with classical parameters $v=(q^6-1)/(q-1)$, $k=(q^5-1)/(q-1)$, $\lambda=(q^4-1)/(q-1)$, and automorphism group $\mathrm{Aut}(G_2(q))$.

1. INTRODUCTION

The classical point-hyperplane design $\mathbf{D}_{n-1}(n,q)$ of an n-dimensional GF(q)-space has classical parameters $v=(q^n-1)/(q-1)$, $k=(q^{n-1}-1)/(q-1)$, $\lambda=(q^{n-2}-1)/(q-1)$. When $n+1$ and q are odd, another symmetric design with these same parameters is the orthogonal design $\mathbf{D}_O(n,q)$ of Higman [Hi]; its points are the singular points x of an $(n+1)$-dimensional orthogonal GF(q)-space and its blocks correspond to the hyperplanes $x^\perp$. These two classes play a special role among the symmetric designs with classical parameters since they admit nonabelian simple point-primitive automorphism groups. In this note we will describe further symmetric designs sharing some of these properties.

[1] This research was supported in part by the National Science Foundation.

0097-3165/02 $35.00

We associate with each generalized hexagon **H** of order (q, q) a symmetric design **D(H)** with parameters $v=(q^6-1)/(q-1)$, $k=(q^5-1)/(q-1)$, $\lambda=(q^4-1)/(q-1)$. Let **H**(q) be the usual generalized hexagon associated with the Chevalley group $G_2(q)$ and let **H**$(q)^*$ be the dual hexagon. If q is not a power of 3 we show that **D(H**$(q)^*$**)** is isomorphic to neither $\mathbf{D}_5(6, q)$ nor $\mathbf{D}_O(6, q)$, and that Aut**D(H**$(q)^*$**)** $\cong$ Aut $G_2(q)$. Moreover, $G_2(q)$ acts primitively on the points and blocks of this design as a rank 4 group, and hence is antiflag transitive.

2. DESIGNS FROM GENERALIZED HEXAGONS

Let **H** be a generalized hexagon of order (q, q) with point set $\mathscr{P}$, line set $\mathscr{L}$ and usual metric d on $\mathscr{P} \cup \mathscr{L}$ (cf. [vM, p. 4]). Each point x determines a partition $\mathscr{P}=\{x\} \cup \Gamma_2(x) \cup \Gamma_4(x) \cup \Gamma_6(x)$ where $\Gamma_i(x)=\{y \in \mathscr{P} \mid d(x, y)=i\}$, $|\Gamma_2(x)|=q^2+q$, $|\Gamma_4(x)|=q^4+q^3$ and $|\Gamma_6(x)|=q^5$.

Set $x^\perp=\{y \in \mathscr{P} \mid d(x, y) \leqslant 4\}$. Denote by **D(H)** the incidence structure with point set $\mathscr{P}$ and block set $\{x^\perp \mid x \in \mathscr{P}\}$. Then:

Proposition 2.1. **D(H)** *is a symmetric* $((q^6-1)/(q-1), (q^5-1)/(q-1), (q^4-1)/(q-1))$*-design, and* $x \mapsto x^\perp$ *is a null polarity of* **D(H)**.

Proof. Generalized hexagons induce rank 3 association schemes on the set of points, using the above partition (see, e.g., [Ma, p. 133]). Checking the parameters of this association scheme we find that $|\Gamma_6(x) \cap \Gamma_6(y)|=q^5-q^4$ for any distinct points x, y. This implies that the map $x \mapsto x^\perp$ is injective hence that the incidence structure complementary to **D(H)** is a symmetric design. It follows that **D(H)** is a symmetric design with the stated parameters. The final assertion is clear from the definition. ▮

The only known generalized hexagons of order (q, q) are the usual one **H**(q) related to $G_2(q)$, as well as its dual **H**$(q)^*$ [Ti] (cf. [vM, Section 2.4]). Therefore, we will focus on these instances of the proposition. First we digress for some general results concerning symmetric designs.

3. PRELIMINARY CHARACTERIZATIONS

For any symmetric design, if x and y are distinct points then their *line* xy is defined to be the intersection of all blocks containing x and y; two points are on exactly one line [DW]. For example, in the preceding section the lines of **H** are also lines of **D(H)**. The Dembowski–Wagner Theorem [DW] characterizes projective spaces as the only symmetric designs such that all lines have size $(v-\lambda)/(k-\lambda)$. The designs **D(H)** provided motivation for variations on their result. For a symmetric design equipped with a null polarity $x \to x^\perp$ call a line *singular* if it contains distinct points x, y such that $y \in x^\perp$, and *nonsingular* otherwise.

THEOREM 3.1 [Ka2]. *Let* **D** *be a symmetric design admitting a null polarity.*

(i) *If all singular lines have size* $(v-\lambda)/(k-\lambda)$ *then* **D** *is either a projective space or an orthogonal design.*

(ii) *If all nonsingular lines have size* $(v-\lambda)/(k-\lambda)$ *then* **D** *is a projective space.*

4. RECOVERING THE GENERALIZED HEXAGON FROM ITS DESIGN

Once again consider the symmetric design $\mathbf{D}=\mathbf{D}(\mathbf{H})$ associated to the generalized hexagon **H**. We wish to recover **H** from **D(H)** using the geometry of the design. However, this is not possible for $\mathbf{D}(\mathbf{H}(q))$ in a canonical manner, since the design arises from large numbers (namely, $|\mathrm{Aut}\,\mathbf{D}(\mathbf{H}(q)) : \mathrm{Aut}\,\mathbf{H}(q)|$) of hexagons:

PROPOSITION 4.1.

(a) *If q is odd then* $\mathbf{D}(\mathbf{H}(q)) \cong \mathbf{D}_O(6, q)$.

(b) *If q is even then* $\mathbf{D}(\mathbf{H}(q)) \simeq \mathbf{D}_S(6, q)$.

(c) *If q is a power of* 3 *then* $\mathbf{D}(\mathbf{H}(q)^*) \cong \mathbf{D}(\mathbf{H}(q))$.

Proof. (a, b) These are clear from the standard embedding of $\mathbf{H}(q)$ into a 7-dimensional orthogonal vector space [Ti] (cf. [Ya, CK, vM, Section 2.4]), in view of the isomorphism $\Omega(7, q) \cong \mathrm{Sp}(6, q)$ when q is even.

(c) $\mathbf{H}(q)^* \cong \mathbf{H}(q)$ for these q. ∎

THEOREM 4.2. *If q is not a power of* 3 *then* $\mathrm{Aut}\,\mathbf{D}(\mathbf{H}(q)^*) \cong \mathrm{Aut}\, G_2(q)$. *In particular,* $\mathbf{D}(\mathbf{H}(q)^*)$ *is isomorphic to neither* $\mathbf{D}_S(6, q)$ *nor* $\mathbf{D}_O(6, q)$.

Proof. $G_2(q)$ acts distance-transitively on the points of $\mathbf{H}(q)^*$): $G_2(q)_x$ is transitive on $\Gamma_i(x)$ for each point x and $i = 2, 4, 6$. Thus, at least one of the following holds: (i) The only $q+1$-point lines of $\mathbf{D}(\mathbf{H}(q)^*)$ are the lines of $\mathbf{H}(q)^*$; (ii) all singular lines have size $q+1$; or (iii) all nonsingular lines have size $q+1$.

In case (i) the hexagon clearly can be recovered from the design in a geometric manner, and hence the assertions of the theorem hold.

It remains to show that (ii) and (iii) do not occur. In either of these situations, we can apply the results of Section 3 and conclude that $\mathbf{D}(\mathbf{H}(q)^*)$ is $\mathbf{D}_5(6, q)$ or $\mathbf{D}_O(6, q)$; moreover, those results imply that (ii) must hold. Then $G_2(q)$ must act on $\mathbf{D}(\mathbf{H}(q)^*)$. However, when q is odd this group does not have a nontrivial projective 6-dimensional GF(q)-representation, so that $\mathbf{D}(\mathbf{H}(q)^*)$ cannot be $\mathbf{D}_5(6, q)$. Consequently, for q even as well as odd, we can identify the points x of $\mathbf{D}(\mathbf{H}(q)^*)$ with the singular points of a 7-dimensional orthogonal geometry in such a way that blocks are identified with the hyperplanes $x^\perp$ of that space.

By [Ya, Theorem 1.1], [Ro] or [CK, Appendix], it follows that $\mathbf{H}(q)^*$ is isomorphic to $\mathbf{H}(q)$. However, $\mathbf{H}(q)$ is not self-dual since q is not a power of 3. Thus, (ii) and (iii) cannot occur. ∎

The group $G_2(q)$can be used to obtain slightly more information about the designs $\mathbf{D}(\mathbf{H}(q)^*)$, namely, the sizes of their lines.

PROPOSITION 4.3. *If q is not a power of 3 then, except for the $q+1$-point lines of $\mathbf{H}(q)^*$, all lines of $\mathbf{D}(\mathbf{H}(q)^*)$ have size 2.*

Proof. Let H denote a subgroup of $G_2(q)$ of order $(q-1)^2$ (a *split torus*; cf. [Ca, Chapter 7]). It is straightforward to use the commutator relations in [Ca, Theorem 12.1.1]) to check that H fixes 6 points of $\mathbf{H}(q)^*$ and that all other point–orbits have length at least $q-1$ (this is even true if $q=2$). If $\mathbf{D}(\mathbf{H}(q)^*)$ has a line of size >2 other than a line of $\mathbf{H}(q)^*$, then it follows that this line has size $q+1$. Now we can proceed exactly as in the proof of the preceding theorem. ∎

5. CONCLUDING REMARKS

1. For q a prime power and $n \geqslant 3$ there are large numbers of (pairwise nonisomorphic) symmetric designs with classical parameters $v=(q^n-1)/(q-1)$, $k=(q^{n-1}-1)/(q-1)$, $\lambda=(q^{n-2}-1)/(q-1)$ [Ju], including large numbers having full automorphism group isomorphic to any given group [Ka1]. However, no design constructed in those papers has an automorphism group transitive on points.

It seems likely that the designs $\mathbf{D}_{n-1}(n, q)$, $\mathbf{D}_O(n, q)$ and $\mathbf{D}(\mathbf{H}(q)^*)$ are the only symmetric designs with classical parameters having an antiflag transitive automorphism group. We have some preliminary results in this direction.

2. The proof of Proposition 4.3 used the commutator relations to study the action of H. A more geometric approach is provided by

[CK, Appendix]. Namely, H lies in a subgroup $SL(3, q)$ preserving a decomposition of an 7-dimensional orthogonal (vector) space as the perpendicular sum of a nonsingular 1-space and two totally singular 3-spaces. Using this description it is easy to check that H fixes 6 totally singular lines and that all other H-orbits of totally singular lines have length $\geqslant q-1$.

3. We determined the automorphism group of $\mathbf{D}(\mathbf{H}(q)^*)$ in Theorem 4.2. More computation within the group $G = G_2(q)$ can be used to obtain a direct proof of Proposition 4.3 and then of Theorem 4.2. We now outline a third type of proof of the proposition and hence of the theorem, this time citing group-theoretic results. We may assume that q is not a power of 3.

Clearly, $G^1 := \mathrm{Aut}G \leqslant A := \mathrm{Aut}\mathbf{D}(\mathbf{H}(q)^*)$. If $G^1 < A$ we deduce from [LPS, p. 133] that there is a subgroup $Y \leqslant A$ with $G < Y$ such that $Y \cong \mathrm{Sp}(6, q)$ or $L_6(q)$ if q is even, while $Y \cong \Omega(7, q)$ if q is odd. If q is even the stabilizer G^1_x in G^1 of a point x of $\mathbf{H}(q)^*$ is therefore the stabilizer of a point in the vector space $V = V(6, q)$ underlying Y. However, by [Li, 2.10] any two nontrivial 6-dimensional $G_2(q)$-modules are algebraically conjugate. In particular, G^1_x must be G^1-conjugate to the stabilizer of a point of $\mathbf{H}(q)$, which is not the case. The case q odd is similar.

Of course, since the proof in [LPS] involves the classification of finite simple groups the preceding is not the simplest approach to the determination of the desired automorphism group.

REFERENCES

[CK] P. J. Cameron and W. M. Kantor, 2-transitive and antiflag transitive collineation groups of finite projective spaces, *J. Algebra* **60** (1979), 384–422.

[Ca] R. Carter, "Simple Groups of Lie Type," Wiley, New York, 1972.

[DW] P. Dembowski and A. Wagner, Some characterizations of finite projective spaces, *Arch. Math.* **11** (1960), 465–469.

[Hi] D. G. Higman, Finite permutation groups of rank three, *Math. Z.* **86** (1964), 145–156.

[Ju] D. Jungnickel, The number of designs with classical parameters grows exponentially, *Geom. Ded.* **16** (1984), 167–178.

[Ka1] W. M. Kantor, Automorphisms and isomorphisms of symmetric and affine designs, *J. Algebraic Combin.* **3** (1994), 307–338.

[Ka2] W. M. Kantor, Variations on the Dembowski–Wagner Theorem, submitted for publication.

[Li] M. W. Liebeck, The affine permutation groups of rank 3, *Proc. London Math. Soc.* **54** (1987), 477–516.

[LPS] M. W. Liebeck, C. E. Praeger, and J. Saxl, A classification of the maximal subgroups of the finite alternating and symmetric groups, *J. Algebra* **111** (1987), 365–383.

[Ma] R. Mathon, 3-class association schemes, *in* "Proc. Conf. on Algebraic Aspects of Combinatorics, Toronto, 1975," pp. 123–155.

[Ro] M. A. Ronan, A geometric characterization of Moufang hexagons, *Invent. Math.* **57** (1980), 227–262.

[Ti] J. Tits, Sur la trialité et certains groupes qui s'en déduisent, *Publ. Math. IHES* **2** (1959), 13–60.

[vM] H. van Maldeghem, "Generalized Polygons," Birkhäuser, Basel/Boston/Berlin, 1998.

[Ya] A. Yanushka, Generalized hexagons of order (t, t), *Israel J. Math.* **23** (1976), 309–324.