Generation of Linear Groups[1]

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1. Introduction

Let G be a *finite*, primitive subgroup of $GL(V) = GL(n, D)$, where V is an n-dimensional vector space over the division ring D. Assume that G is generated by "nice" transformations. The problem is then to try to determine (up to $GL(V)$-conjugacy) all possibilities for G. Of course, this problem is very vague. But it is a classical one, going back 150 years, and yet very much alive today. The purpose of this paper is to discuss both old and new results in this area, and in particular to indicate some of its history. Our emphasis will be on especially geometric situations, rather than on representation-theoretic ones.

For small n, all transformations may be considered "nice" (Sections 2 and 4). For general n, the nicest transformations are reflections and transvections (or, projectively, homologies and elations); these occupy Sections 3 and 5. Finally, Section 6 touches on several other types of "nice" transformations.

We will generally regard as equivalent the study of subgroups of $GL(n, D)$ and of the projective group $PGL(n, D)$. It should, however, be realized that this point of view was occasionally not taken by some of the authors cited here.

In general, we will not list the groups in the classifications discussed; nor will we discuss further properties of the groups obtained.

Further historical information may be found in Wiman (1899b) and van der Waerden (1935).

2. Characteristic 0: Small Dimensions

While the subject of this paper began in the case of finite D, we will start with the possibly more familiar characteristic 0 case. In this section, D will be

[1] Supported in part by NSF Grant MCS-7903130.
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commutative of characteristic 0—in which case we may take $D = \mathbb{C}$—and n will be small. By a fundamental result of Jordan (1878, 1879), for each n the number of types of primitive subgroups of $SL(n, \mathbb{C})$ is finite.

All finite subgroups of $SL(2, \mathbb{C})$ were first determined by Klein in 1874 (Klein (1876, 1884)). His method was very geometric, based upon regarding the extended complex plane as a sphere in $\mathbb{R}^3$. Of course, the groups he found all arise from regular polygons and regular polyhedra.

Jordan, who had been working on $SL(2, \mathbb{C})$, turned to $SL(3, \mathbb{C})$ (Jordan (1878)). However, he missed two examples (later found by Klein (1879) and Valentiner (1889)). His approach was not at all geometric. He derived information about G by a case-by-case analysis of a diophantine equation he had used successfully in the proof of his general finiteness theorem. (This equation arises by expressing $|G|$ as a sum in terms of the orders of suitable—and especially, maximal—abelian subgroups of G and of the indices of their normalizers, great care being taken with intersections of pairs of such subgroups.) He used the same methods soon afterwards (Jordan (1879)) in order to (attempt to) correct his previous work on $SL(3, \mathbb{C})$, and in order to obtain very preliminary results concerning $SL(4, \mathbb{C})$. His diophantine approach was later used a number of times, especially in the case of finite fields (Moore (1904), Wiman (1899a), Dickson (1900), Mitchell (1911a, 1913), Huppert (1967)).

Valentiner (1889) devised a similar diophantine method in his attempt at $SL(3, \mathbb{C})$. In addition, he proceeded somewhat geometrically, but erred in his treatment of homologies of order 3 (Mitchell (1911b)), thereby missing one example. (He was apparently unaware of Jordan's work on the same problem, where this example is listed.) Valentiner's treatment seems to have otherwise been correct: Wiman (1896) stated that Valentiner's error was easily corrected, and that all examples were known. For further historical discussion up to this point, as well as for properties of these groups, see Wiman (1899b).

Blichfeldt (1904, 1907) was the first to publish a complete proof for $SL(3, \mathbb{C})$. His methods were nongeometric: they involved a careful analysis of eigenvalues in order to obtain precise information concerning $|G|$. A purely geometric proof was later obtained by Mitchell (1911a). In fact, since it is easy to show that a primitive subgroup of $PSL(3, \mathbb{C})$ contains homologies (compare Mitchell (1911a), p. 215), a geometric proof is implicitly contained in Bagnera (1905); for the same reason, Mitchell's proof depends upon homologies (cf. Section 3).

Eigenvalue and order considerations also dominate the determination by Blichfeldt (1905) (also 1917) of all finite primitive subgroups of $SL(4, \mathbb{C})$. At about the same time, Bagnera (1905) gave a geometric solution to this problem when G contains homologies; the case when G does not contain homologies was handled later by Mitchell (1913), thereby providing an alternative, geometric proof of Blichfeldt's result.

At this point, the subject seems to have died, probably because much more sophisticated methods were needed. It was finally revived again by Brauer (1967), who handled $SL(5, \mathbb{C})$. The cases $n = 6$, 7, 8, and 9 have now been completed, by Lindsay (1971), Wales (1969, 1970), Doro (1975), Huffman and Wales (1976, 1978), and Feit (1976). In these results, geometry essentially

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3. Characteristic 0: Reflections

Recall that a *reflection* is a diagonalizable transformation having eigenvalue 1 with multiplicity $n-1$. The corresponding eigenspace is its *axis*; the remaining 1-dimensional eigenspace is its *center*. A *homology* is just a reflection viewed projectively (i.e., as acting on $PG(n-1, D)$). Classification problems concerning reflections or homologies are thus essentially the same, and will generally be identified.

Finite subgroups of $GL(n,\mathbb{R})$ generated by reflections are a very familiar topic. For a discussion of them and their history, we defer to Coxeter (1948) and Bourbaki (1968). However, it is worth mentioning that the classification and study of these groups occupy a far more central role in mathematics than the other groups discussed in this survey. They are the crystals (or rather, "apartments") from which Tits' theory of buildings grows (Tits (1974), Carter (1972)), and hence are central in the theories of algebraic groups (Tits (1966)) and of finite groups (Chevalley (1955), Carter (1972)). Further incredibly varied but fundamental occurrences of them are discussed at length in Hazewinkel et al. (1977).

The determination of all finite primitive subgroups of $GL(n,\mathbb{C})$ generated by reflections is due primarily to Mitchell (1914a). Namely, he dealt with the cases $n \geqslant 5$, the smaller values of n having been handled earlier (as described in Section 2). His method was short, elegant, and very geometric. It involved building up groups, homology by homology and dimension by dimension. Namely, suppose that W is a subspace of V, spanned by some of the homology centers for G, and for which the induced group generated by these homologies is known—and, hopefully, primitive. Mitchell picked a homology h moving W, with center c, and studied the group induced on $\langle W, c\rangle$. (Since a homology fixes every subspace containing its center, both the known group and h send $\langle W, c\rangle$ to itself.)

However, Mitchell's result apparently went largely unnoticed. He was clearly far ahead of his time: he handled the complex case several years before all real reflection groups were independently determined by Cartan and Coxeter (cf. Coxeter (1948, p. 209), and Bourbaki (1968, p. 237)). Only very recently has another complete proof of his result appeared (Cohen (1976)). Important special cases have, however, been re-proved (Shepard (1952, 1953); Coxeter (1957), (1974)); namely, those leading to regular complex polytopes.

Shephard and Todd (1954) took the (projective) groups generated by homologies obtained by Klein (1876), Blichfeldt (1904, 1907), Bagnera (1905), and Mitchell (1914a, b), and listed all complex reflection groups giving rise to them. The case $n \geqslant 3$ is implicit in the above papers (and is freely used in Mitchell's proof); the case $n=2$ is more involved. This list will not be reproduced here. Instead, we will simply make a few comments about the largest example which is not already a real reflection group.

A group $G = 6 \cdot P\Omega^-(6,3)\cdot 2$, having $|Z(G)| = 6$, $|G : G'| = 2$, and $G'/Z(G) \cong P\Omega^-(6,3)$, arises as a subgroup of $GL(6,\mathbb{C})$ generated by involutory reflections. It was discovered by Mitchell (1914a), who wrote down coordinates for its reflecting hyperplanes. Geometric properties of the action on the corresponding projective space $PG(5,\mathbb{C})$ were studied by Hamill (1951) and Hartley (1950). Its reflection centers (dual to the reflecting hyperplanes) determine the $\mathbb{Z}[\omega]$-lattice Λ of Coxeter and Todd (1953) (where ω is a primitive cube root of unity). This lattice consists of all $(x_i) \in \mathbb{Z}[\omega]^6$ such that $\sum_1^6 x_i \equiv 0 \pmod 3$ and $x_i \equiv x_j \pmod \theta$ for all i, j (where $\theta = \omega - \omega^2$ satisfies $\theta^2 = -3$); Λ is equipped with the usual hermitian inner product inherited from $\mathbb{C}^6$. Its automorphism group is G, generated by the reflections in $GL(6,\mathbb{C})$ preserving Λ; these are the reflections with centers $\langle\lambda\rangle$ for $\lambda \in \Lambda$ of norm 6. This group induces $\Omega^-(6,3)\cdot 2$ on $\Lambda/\theta\Lambda$, where $\Lambda/\theta\Lambda$ is the natural $GF(3)$-module for $O^-(6,3)$. The 126 reflections in G induce 126 reflections of the orthogonal space $\Lambda/\theta\Lambda$. The remaining 126 reflections of that space are induced by using semilinear automorphisms of Λ; for example, $-cr$ induces one of them, where c denotes complex conjugation on Λ, while r is the reflection with center $\langle(1,1,1,1,1,1)\rangle$. On the other hand, the hermitian product on Λ induces one on the $GF(4)$-space $\Lambda/2\Lambda$, and reflections in G induce 126 transvections (defined in Section 5) belonging to $SU(6,2)$. This produces an embedding $P\Omega^-(6,3)\cdot 2 < PSU(6,2)$, which is crucial to the existence of the sporadic finite simple groups found by Fischer (1969). Also, the lattice $\Lambda \oplus \Lambda$ is a sublattice of the Leech $\mathbb{Z}[\omega]$-lattice, described in Conway (1971). Similarly, the direct sum of three copies of the 8-dimensional real lattice of type E_8 is a sublattice of the Leech lattice itself (Conway (1971)); while the corresponding real reflection group, when embedded in $O^+(8,3)$, also plays a significant role in Fischer's constructions.

The study of small-dimensional complex groups, and of large-dimensional groups generated by reflections, seems to have (temporarily) ended with Blichfeldt (1917) and Mitchell (1914a, b). Mitchell's attitude towards this is indicated on pp. 596–7 of Mitchell (1935). First he states that "comparatively few groups of interest appear to be known in more than four variables." This leads to a discussion of work of Burnside (1912) concerning real reflection groups. Mitchell then turns to his own work on complex reflection groups: "In spite of the more general character of this problem as compared with that solved by Burnside, no restrictions being placed on the character of the coefficients, the results were chiefly negative." Only one new example arose (the 6-dimensional one just discussed). Thus, Mitchell was looking for new groups, or at least new linear groups, and was not entirely happy with the outcome of this work.

It is unfortunate, both for geometry and group theory, that Mitchell (or someone else of his generation) did not pursue reflections further. Certainly, if D is commutative of characteristic 0, then D may be assumed to be a subfield of $\mathbb{C}$. However, reflection groups over the quaternions $\mathbb{H}$ do indeed yield new examples. One 3-dimensional example is (projectively) a simple group discovered in 1967. Its discovery 50 years earlier might have revived the then nearly dead theory of finite groups.

The determination of all finite primitive subgroups of $GL(n,\mathbb{H})$ generated by reflections was made by Cohen (1980), although some of this had been done earlier by Wales and Conway. The groups G obtained which are not complex reflection groups can be described as follows, if $n \geqslant 3$:

(i) $n = 3$, $G = Z_2 \times PSU(3,3)$;
(ii) $n = 3$, $G = 2 \cdot HJ$ (where HJ denotes the Hall–Janko simple group, predicted by Janko in 1967 and constructed by Hall as a permutation group of degree 100 on the cosets of a subgroup $PSU(3,3)$; cf. Hall and Wales (1968);
(iii) $n = 4$, $G/Z(G)$ has an elementary abelian normal subgroup of order 2^6, modulo which it is one of 3 subgroups of $\Omega^-(6,2)$ (note the similarity to some 4-dimensional complex groups);
(iv) $n = 4$, $G/Z(G) \cong (A_5 \times A_5 \times A_5) \rtimes S_3$ (a wreathed product; compare the situation $(A_5 \times A_5) \rtimes S_2$ for the real reflection group [3, 3, 5]); and
(v) $G = Z_2 \times PSU(5,2)$.

In each case, all reflections turn out to be involutory. Tits has shown that example (ii) is related to a quaternionic version of the real Leech lattice.

Cohen's proof is definitely nongeometric. Quaternionic n-space can be regarded as complex $2n$-space (in many ways). When this is done, quaternionic reflections become complex transformations having a $(2n-2)$-dimensional eigenspace. Results of Huffman and Wales (Huffman (1975); Huffman and Wales (1975); Wales (1978)), to be discussed soon, then provide a list of complex groups; these must be checked to see which arise from quaternionic groups.

It would be desirable to have a new geometric proof of Cohen's result. The present proof is not elegant, using machinery of an overly sophisticated sort. A new proof would presumably proceed along the lines of Mitchell's approach. The case $n = 2$ merely requires knowledge of the finite subgroups of $SL(4,\mathbb{C})$. The case $n = 3$ is probably the hardest and most interesting one, in view of the examples. Starting from these cases, Mitchell's approach should have a good chance of success.

In the papers just cited, Huffman and Wales extended Mitchell's work in quite a different direction. They determined all finite primitive subgroups of $GL(n,\mathbb{C})$ which are generated by transformations having $(n-2)$-dimensional eigenspaces. The resulting list is too long to reproduce here, but is probably worthy of geometric investigation. It may not be possible to give a direct proof of their result. Their proof relies very heavily on representation theory (ordinary and modular), and on very deep simple group classification theorems. Little geometry is involved. It is precisely for this reason that an alternative approach is needed to Cohen's quaternionic results.

However, there is an obvious advantage to applying group-theoretic classification theorems in geometry: results can be obtained which may otherwise be difficult to prove, or which may later be proved more elegantly. For example, consider the problem of determining all finite primitive reflection groups G in $GL(n,D)$, for D an arbitrary noncommutative division ring of characteristic 0. If $n = 1$, this is just the famous problem solved by Amitsur (1955) (and independently and almost simultaneously by J. A. Green). If $n = 2$ and G is solvable, the problem seems to involve even more difficult number theory than Amitsur used.

But if $n \geqslant 3$, and if simple group classification theorems are thrown at the problem, no new nonsolvable examples arise. Similarly, the Cayley–Moufang projective plane appears not to admit any new examples of finite groups, generated by involutory reflections, which fix no point, line, triangle, or proper subplane, other than ${}^3D_4(2)$.

We have only been discussing the classification of reflection groups. There is, of course, a large body of literature concerning their properties. Their invariants have been of interest for a century (see, e.g., Klein (1876, 1884), and Shephard and Todd (1954), and the papers by Hiller and Solomon in these Proceedings). So have their associated polytopes in the real and complex cases (Coxeter (1948, 1957, 1974); Shephard (1952, 1953)). The case of quaternionic polytopes has recently been begun by Hoggar (1978) (see also his paper in these Proceedings). For remarkable extremal properties of real, complex, and quaternionic examples, see Delsarte, Goethals and Seidel (1975, 1977), Hoggar (1978), and Odlyzko and Sloane (1979).

4. Finite D: Small Dimensions

The detailed study of the subgroups of $PSL(2,D)$ was begun by Galois in 1832 with the case of a prime field D (cf. Galois (1846), pp. 411–412, 443–444). For prime q, all subgroups of $PSL(2,q)$ were first determined by Gierster (1881). Burnside (1894) worked on the case of arbitrary q. Finally, all subgroups of $PSL(2,q)$ were determined for all q independently by Moore (1904) and Wiman (1899a). We refer to Kantor (1979b) and references given there for further historical remarks concerning 2-dimensional groups.

The group $PSL(3,q)$ brings us back to Mitchell. The first attempt at determining its subgroups was made by Burnside (1895) in case q and $(q^2+q+1)/(3,q+1)$ are both prime; but he missed the groups $PSO(3,q)$. Dickson (1905) later enumerated all subgroups of order divisible by q, when q is prime, using an explicit knowledge of all conjugacy classes of q-groups. Both authors relied on group theory and matrices, not on geometry. Veblen suggested to his student Mitchell that he provide a geometric solution to the problem for $PSL(3,5)$ (where, incidentally, q^2+q+1 is prime). Mitchell solved the problem for $PSL(3,q)$, first for odd prime q, then for arbitrary odd q in his thesis "The subgroups of the linear group $LF(3,p^n)$," written in 1910; the solution appears in Mitchell (1911a). (Another student of Veblen's, U. G. Mitchell, determined the subgroups of $PSL(3,4)$ in his thesis entitled "Geometry and collineation groups of the finite projective plane $PG(2,2^2)$," also written in 1910.) H. H. Mitchell went even further in his paper: he dealt with $PSL(3,\mathbb{C})$ at the same time as $PSL(3,q)$. His approach was very geometric, and highly original. (A very different approach, based on modular characters and simple group classification theorems, was given by Bloom (1967).) It should, in fact, be noted that Mitchell solved problems which Jordan (1878, 1879), Valentiner (1889), Burnside (1895), and Dickson (1905) could not. The maximal subgroups of $PSL(3,q)$, q even, were later determined by Hartley (1926) in his thesis written under Mitchell. By

Mitchell (1911a), $|G|$ must be even here, so Hartley naturally concentrated on the elations G must contain (cf. Section 5).

Mitchell's only other major papers on linear groups were Mitchell (1913), where all subgroups of $PGL(4,\mathbb{C})$ and $PGL(4,q)$ are determined which do not contain nontrivial homologies and have order not divisible by the characteristic of the field; Mitchell (1914a), which was discussed in Section 2; and Mitchell (1914b), in which all maximal subgroups of the symplectic groups $Sp(4,q)$ were found for odd q. All four papers rely heavily on geometry. The most important ones are certainly the ones on reflection groups and $PSL(3,q)$. The work of Mitchell and Hartley on $PSL(3,q)$ has been quoted often in recent papers on finite groups, besides providing some motivation for Piper's work on elations of finite projective planes (Piper (1965, 1966b)).

The groups $PSL(n,q)$, $n=4$ or 5, have been the object of several recent papers. Mwene (1976) and Wagner (1979) enumerated all maximal subgroups when q is even and n is 4 and 5, respectively. The same was done, independently, by Zalesskii (1977). Zalesskii and Suprenenko (1978) handled the case $PSL(4,q)$ when the prime p dividing q is greater than 5, and Mwene (1980) discussed the general case for odd characteristic. $PSL(5,q)$ was handled by Zalesskii (1976) for $p>5$, and completed for $p \geqslant 3$ by DiMartino and Wagner (1981). All these papers rely heavily on modular representation theory and simple group classification theorems. See Kantor and Liebler (1982) for further discussion and applications of these results.

5. Finite D: Homologies and Elations

Mitchell (1914a) observed that his work on complex groups generated by homologies applied equally well when the field was $GF(q)$, so long as q is relatively prime to the order of the group. When this condition fails, so does complete reducibility, and the problem becomes considerably harder. As a further indication of its difficulty, note that Mitchell's problem turned out to be a finite one: only finitely many primitive examples exist. However, when $D = GF(q)$ and $q>2$, infinitely many examples arise, such as orthogonal groups, unitary groups, and $PGL(n,q)$ itself. In addition, complex examples produce examples for suitable odd q, simply by passing modulo a suitable prime ideal. Of course, all of the above remarks apply to Section 4 as well.

Primitive subgroups of $PGL(n,q)$ containing a homology of order greater than 2 were determined independently by Wagner (1978) and by Zalesskii and Serezkin (1977). Homologies of order 2 were handled by Serezkin (1976) when q is not a power of 3 or 5. The general case of groups containing involutory homologies was settled by Wagner (1980–1981). All of these papers are highly geometric. The general case was also dealt with independently and nongeometrically by Zalesskii and Serezkin (1980).

Wagner's approach is based on that of Mitchell (1914a). It is direct and reasonably elementary (but long). More than half of the work is devoted to fields of characteristic 3 or 5. The results may be summarized as follows.

Suppose that G contains involutory homologies, but no homologies of higher order and no nontrivial elations (defined below). Then either

(i) $G \trianglerighteq P\Omega^{\pm}(n,q')$ with $GF(q') \subseteq GF(q)$;
(ii) $G = S_{n+2}$ and $(q, n+2) \neq 1$;
(iii) G arises from a complex reflection group; or
(iv) $G = PSL(3,4) \cdot 2$, $n = 4$, and $GF(9) \subseteq GF(q)$.

Example (iv) arises from the embedding $PSL(3,4) \cdot 2 < PSU(4,3) \cdot 2$, which in turn arises from the complex 6-dimensional reflection group discussed in Section 3. The embedding $PSL(3,4) < PSU(4,3)$ was discovered by Hartley (1950) by considering the action of that reflection group on $PG(5,\mathbb{C})$. An alternative proof can be given, by observing that $SL(3,4)$ is induced on any totally isotropic 3-space of the unitary space $\Lambda/2\Lambda$ which is fixed by none of the transvections in the group. This embedding is the basis for the construction by McLaughlin (1969) of his sporadic simple group.

Homologies are not the only collineations inducing the identity on a hyperplane of a projective space. The other type of collineations behaving in this manner are the *elations*. They have order 1 or p if D has characteristic $p \neq 0$. The corresponding linear transformations are *transvections*; such a transformation t satisfies $(t-1)^2 = 0$ and $\dim V(t-1) \leqslant 1$. Then, with respect to some basis, t has the form

$$t = \begin{bmatrix} 1 & 0 & & \alpha \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \quad \text{for some } \alpha \in D;$$

if α is allowed to be arbitrary, then the resulting transvections form a group $\cong D^{+}$, called a *root group*. (This is a special case of root groups of Chevalley groups; cf. Carter (1972).)

McLaughlin (1967, 1969a) determined all irreducible subgroups of $GL(n,D)$ generated by root groups, for any field D. His approach is elegant and geometric.

The primitive subgroups of $PSL(n,q)$ generated by elations have also been determined, primarily by Piper (1966b, 1968, 1973) and Wagner (1974) (and, independently, by Zalesskii and Serezkin (1976) for odd q). Their arguments are beautifully geometric. Unfortunately, in one characteristic 2 situation simple group classifications were also used (Kantor (1979a)). For $n \geqslant 4$, the possibilities are as follows:

(i) $PSL(n,q')$, $PSp(n,q')$, and $PSU(n,q')$, where $GF(q') \subseteq GF(q)$;
(ii) $PO^{\pm}(n,q')$, where q' is even and $GF(q') \subseteq GF(q)$;
(iii) S_{n+2}, where n and q are even; and
(iv) $P\Omega^{-}(6,3) \cdot 2$, where $n = 6$ and $GF(4) \subseteq GF(q)$.

Of course, example (iv) arises from Mitchell's 6-dimensional complex reflection group. An entirely geometric proof of the above result would again be desirable.

Elations appear in several situations. Ever since Galois, they have been involved in the proof of the simplicity of linear groups—not just of $PSL(n,q)$, but also of $PSp(2n,q)$ and $PSU(n,q)$ (Jordan (1870), Dickson (1900), Huppert (1967), as well as implicitly in Carter (1972)). Elations and homologies were used throughout the study of subgroups of $PSL(3,q)$ by Mitchell (1911a) and Hartley (1926). Elations were equally important for $PSL(4,q)$ and $PSL(5,q)$; for example, if q is even, then the Sylow 2-subgroups of a subgroup of $PSL(5,q)$ containing no nontrivial elations have nilpotence class at most 2, a fact which was crucial for Mwene (1976), Wagner (1979), and Zalesskii (1977). Elations also arose in the determination of the 2-transitive permutation representations of $PSL(n,q)$, $PSp(2n,q)$, and $PSU(n,q)$ (Curtis, Kantor, and Seitz (1976)); in particular, McLaughlin's result was essential for $PSp(2n,2)$. Elations (and involutory reflections) arise throughout the classification of Fischer (1969); and Fischer's work was, in fact, used at one point in the determination of the primitive groups generated by elations. The latter determination was fundamental in bounding from below the degree (among other things) of a primitive permutation representation of $PSL(n,q)$, $PSp(2n,q)$, or $PSU(n,q)$ (Patton (1972), Cooperstein (1978), Kantor (1979b)).

6. Other Transformations

We conclude with a brief discussion of subgroups G of $GL(V) = GL(n,q)$ generated by other "nice" types of transformations.

(i) Call $t \in GL(V)$ *quadratic* if $(t-1)^2 = 0$. Clearly, $|t|$ is 1 or the prime p dividing q. Transvections are quadratic, and if $p = 2$ then so are all involutions. If t is quadratic and $t \neq 1$, then the subspace $C_V(t)$ of fixed vectors contains the intersection $[V,t] = \{vt - v \mid v \in V\}$ of all fixed hyperplanes. Thus, quadratic transformations can be regarded as generalizations of transvections.

Thompson (1970) classified all irreducible groups generated by quadratic transformations if $p > 3$, at the same time determining all possible modules for each group obtained. The groups are $SL(n,q')$, $Sp(n,q')$, $SU(n,q')$, $\Omega^{\pm}(n,q')$, $G_2(q')$, ${}^3D_4(q')$, $F_4(q')$, $E_6(q')$, ${}^2E_6(q')$, and $E_7(q')$, where $q' \mid q$. (The last six classes of groups are defined in Carter (1972): they are Chevalley and twisted groups.) Some sporadic simple groups arise when $p = 3$; this case has been the subject of a great deal of work by Ho (cf. Ho (1976) and the references given there). Thompson's theorem provided part of the impetus for the remarkable result of Aschbacher (1977) (where no module is present). The latter result to a certain extent supersedes Thompson's, and was a main tool in Ho (1976).

(ii) Dempwolff (1978, 1979) has classified all irreducible subgroups of $SL(n,2)$ generated by involutions t for which $\dim C_V(t) = n-2$. His proof uses simple group classification theorems.

(iii) Kantor (1979a) determined all irreducible subgroups of orthogonal groups $\Omega^{\pm}(n,q)$ which are generated by "long root elements." These are analogues of transvections, provided by the theory of Chevalley groups. While they are quadratic transformations, it is the characteristic 2 case that provides the most interesting examples.

The corresponding type of problem for all other Chevalley groups has been settled by Cooperstein (1979, 1981).

Of greater importance is the work recently begun by Seitz concerning the structure of subgroups of Chevalley groups. When specialized to the case of $SL(n,q)$, one of the preliminary applications of his methods (Seitz (1979)) is the determination of all subgroups of $SL(n,q)$ containing all diagonal matrices when $q > 11$ and q is odd. His methods depend upon algebraic groups, not geometry. Further results on generation of yet another type are found in Seitz (1982).

(iv) *Singer cycles* are elements of $GL(n,q)$ of order $q^n - 1$. Their geometric significance was first noticed by Singer (1938). They arise in the special case $k = 1$ of the following construction.

Let $k \mid n$, and write $s = n/k$. Then a k-dimensional vector space over $GF(q^s)$ is also an n-dimensional vector space over $GF(q)$. Thus, $GL(k,q^s) \leqslant GL(n,q)$. In particular, $GF(q^n)^* \cong GL(1,q^n) \leqslant GL(n,q)$.

Kantor (1980) showed that any subgroup of $GL(n,q)$ generated by Singer cycles is a group $GL(k,q^s)$ (for some k and $s = n/k$) obtained in the above manner. This time, simple group classification theorems are in no way involved in the proof. The proof is geometric, and is based upon the determination (geometrically) of all collineation groups acting 2-transitively on the points of a finite projective space (Cameron and Kantor (1979)).

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