A brief graph-theoretic introduction to buildings

William M. Kantor*

University of Oregon

1. Introduction

Buildings were introduced by J. Tits in order to encode various fundamental geometric and combinatorial properties of simple Lie, algebraic and finite groups (cf. [8]). More recently, he generalized the notion of building in [9], partly in response to the discovery of interesting "near-buildings" (e.g., in [4]). The purpose of this paper is to survey this new approach, with an emphasis on its graph-theoretic aspects.

Every rank r building is a highly structured r-partite graph $(r \geq 2)$ with the property that, for $r > 2$, all neighborhoods of vertices are rank $r - 1$ buildings. As we will see, there are non-buildings that also share this recursive property. First, in § 2 we discuss rank 2 buildings, and use them in § 3 to define GABs ("graphs that are almost buildings"). Four types of examples are given. In § 4 covers of GABs are introduced and used to define buildings. Finally, § 5 contains further examples and open problems.

GABs are natural from a graph-theoretic point of view, especially in the bipartite case (rank $r = 2$). They are somewhat rare, but large enough quantities exist to make

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their study and classification both interesting and difficult. One of their many interesting aspects is that the study of finite GABs forces the consideration of infinite ones.

2. Rank 2 buildings

A rank 2 building is a connected bipartite graph Γ having diameter n and girth $2n$ for some integer $n \geq 2$, and such that all vertices have valence ≥ 3. (Such a graph is also called a generalized n-gon, since the incidence graph of an ordinary n-gon shares the defining properties other than valence.)

The case $n = 2$ is equivalent to that of complete bipartite graphs in which both parts have size ≥ 3.

The case $n = 3$ is precisely the same as that of the incidence graph of a projective plane. Thus, we already see that some buildings are very familiar combinatorial objects.

There are several types of examples known when $n = 4$. For example, let K be a field, and for (α_i), $(\beta_i) \in K^4$ write $((\alpha_i), (\beta_i)) = \alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_3\beta_4 - \alpha_4\beta_3$. The vertices of Γ are the 1-spaces of K^4, and also those 2-spaces U such that $(U, U) = 0$. Adjacency is just inclusion.

When $K = GF(2)$ this is an especially familiar example. It has 30 vertices that can be identified with all the 2-sets of $S = \{1, 2, 3, 4, 5, 6\}$ and all the partitions 2/2/2 of S. Adjacency is the obvious one

(essentially inclusion).

We are not assuming finiteness in this paper, for reasons that will become clear in § 4. However, the most studied rank 2 buildings are the finite ones. Assume that Γ is finite. The Feit-Higman theorem (see [1, Ch. 23; 2]) asserts that $n = 2, 3, 4, 6$ or 8. Let $n = 4, 6$ or 8. If V_1 and V_2 are the two parts of our bipartite graph then all vertices in V_1 have the same valence $s + 1$, and all in V_2 have the same valence $t + 1$. The integers s and t are the parameters of Γ. Several restrictions on them are known. For example, $t \leq s^2$ when $n = 4$ or 8, while $t \leq s^3$ when $n = 6$. Also, st is a square when $n = 6$, and $2st$ is a square when $n = 8$.

There are rank 2 buildings known in the following situations: $n = 4$, $\{s, t\} = \{q, q\}, \{q, q^2\}, \{q^2, q^3\}, \{q-1, q+1\}$; $n = 6$, $\{s, t\} = \{q, q\}, \{q, q^3\}$; $n = 8$, $\{s, t\} = \{q, q^2\}$. Here, q is any prime power, except that q must have the form 2^{2e+1} in the $n = 8$ examples. When $n = 6$ or 8, only one example is known for each q, and it seems very plausible that these are the only possibilities for these values of n. On the other hand, there are a few possibilities known for $n = 4$ and $\{s, t\} = \{q, q\}$ or $\{q-1, q+1\}$, q even; and two possibilities are known when $n = 4$, $\{s, t\} = \{q, q^2\}$, $q \equiv 2 \pmod 3$, $q > 2$.

In other words, finite examples with $n > 3$ are relatively scarce -- much more so than finite projective planes.

3. GABs

Let $r \geq 2$. By a labeled K_r we mean the complete graph D on r vertices $1, \cdots, r$ (say) with each edge $\{i, j\}$ labeled by an integer $D(i, j) = D(j, i) \geq 2$.

A rank r GAB with diagram D is defined recursively as follows. First of all, it is a connected r-partite graph Γ with parts $V_1, \cdots, V_r$. When $r = 2$, Γ is a rank 2 building (§ 2) of diameter $D(1, 2)$. When $r > 2$, for any vertex v, in V_i, say, the neighborhood $\Gamma(v) = \{$vertices joined to $v\}$ is a GAB whose diagram is obtained from D by deleting vertex i and all edges through it.

Thus, if $i \neq j$ and X is any $r-2$-clique in Γ not involving vertices from $V_i \cup V_j$, then its neighborhood $\Gamma(X) = \cap\{\Gamma(x) \mid x \in X\}$ has diagram •$\overset{D(i, j)}{\text{——}}$•.

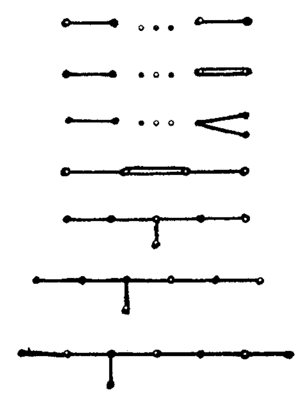
The abbreviated diagram of Γ is obtained from D as follows: every edge •$\overset{2}{\text{—}}$•, •$\overset{3}{\text{—}}$• or •$\overset{4}{\text{—}}$• is replaced by a non-edge • •, an unlabeled adge •—•, or a double edge •═•, respectively. The most important GABs have diagrams with only these particular labels (see the examples indicated below). Moreover, we now have the notation of a connected diagram. It is easy to see that any GAB with disconnected diagram can be obtained by glueing together smaller rank GABs and null graphs.

Example 1. Let V be an n-dimensional vector space over a field K. Let Γ be the $(n-1)$-partite graph whose vertices are all the proper subspaces of V, with adjacency being inclusion. Then Γ is a rank $n-1$

GAB with diagram •—• ... •—• (a path). An $n-3$-clique X is just an increasing sequence of subspaces, and $\Gamma(X)$ is either complete bipartite (with parts of size $|K| + 1$) or the projective plane over K.

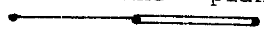
Example 2 [7]. Let V be a $2r$-dimensional vector space over a field K. Let (u, v) be a nonsingular alternating bilinear form, obtained as in § 2 when $2r = 4$. Then let Γ consist of all proper subspaces U such $(U, U) = 0$, again with adjacency being inclusion. This produces a rank r GAB with diagram •—• ... •═• (a path of r vertices with exactly one double edge, at the end of the path).

Examples 1 and 2 belong to a large class of examples [7]. If D is the diagram of a crystallographic semi-regular polytope in $\mathbb{R}^r$, $r > 2$, and K is any field (perfect if the characteristic is 2) then there is at least one GAB with diagram D "coordinatized" by K (somewhat as in the preceding examples). The relevant diagrams are as follows.



The finite GABs with these diagrams and no double edges are all known. No corresponding result is known in the double edge case.

Example 3. The only "peculiar" finite  known is obtained as follows. There are 7 points 1, 2, 3, 4, 5, 6, 7, and 35 lines, all $\binom{7}{3}$ subsets of 3 points. Finally, there are 15 planes, each of which has the structure of a PG(2 , 2) consisting of all 7 points and 7 of the 35 lines.



Each PG(2 , 2) structure uniquely determines 14 other PG(2 , 2)s such that any two of the 1 + 14 PG(2 , 2)s have exactly one common line. The vertices of Γ consist of all of the points, lines and planes, and adjacence is just inclusion. (For example, every point is joined to every plane.) Note that the subgraph consisting of all lines and planes is isomorphic to the incidence graph of all lines and planes of PG(3 , 2).

For each vertex v, $\Gamma(v)$ is a PG(2 , 2), $K_{3,3}$ or 30-vertex graph in § 2, according to whether v is a plane, line or point.

Example 4. Examples of GABs  [4]. Let $M = \{(\alpha_i) \in Q^6 \mid$ denominator of each α_i is a power of $2\}$. An orthogonal basis of M is any sextuple $e_1,\cdots,e_6 \in M$ such that $e_i \cdot e_j = 0$ $\forall i \neq j$ (the usual dot product) and every element of M can be written $\Sigma\beta_i e_i$, where the denominator of each β_i is a power of 2. A nice basis is an orthogonal one such that $e_i \cdot e_j = 2\delta_{ij}$

for all i , j. In this case there are 15 corresponding nice bases, such as

$$f_1 , f_2 = e_1 \pm e_2 \; ; \; f_3 , f_4 = e_3 \pm e_4 \; ; \; f_5 , f_6 = e_5 \pm e_6 .$$

Write $v_1 = \{\langle e_1\rangle,\cdots,\langle e_6\rangle\}$

$v_3 = \{\langle f_1\rangle,\cdots,\langle f_6\rangle\}$

$v_2 = \{\langle e_1\rangle , \langle e_2\rangle , \langle f_1\rangle , \langle f_2\rangle , \langle e_i \pm e_j\rangle , \langle f_i \pm f_j\rangle ,$ where $j > i \geq 3\}$. Then each vertex of Γ looks like v_1, v_2 or v_3, and each edge has the form $\{v_1 , v_2\}$, $\{v_2 , v_3\}$, or $\{v_3 , v_1\}$, for some orthogonal basis $e_1,\cdots,e_6$ and some permutation of $1,\cdots,6$.

This produces a GAB  each of whose neighborhoods is either a 30-vertex example (§ 2) or $K_{3,3}$.

Now let m be any odd integer > 1. Read all of the above mod m. This corresponds to a graph homomorphism $\Gamma \to \Gamma(\text{mod } m)$ onto another GAB called $\Gamma(\text{mod } m)$ with diagram , but now we have finite GABs. Moreover, if $1 < k \mid m$ then there is a homomorphism $\Gamma(\text{mod } k) \to \Gamma(\text{mod } m)$.

All of these homomorphisms have the property that they are locally isomorphisms: each neighborhood in Γ is mapped isomorphically onto a neighborhood in $\Gamma(\text{mod } m)$.

4. Covers and buildings

Let Γ' and Γ be GABs with the same diagram D and rank $r > 2$, and partitions $V_1,\cdots,V_r$ and $V'_1,\cdots,V'_r$. A cover $\pi:\Gamma' \to \Gamma$ is an adjacency-preserving map from Γ' onto Γ, sending V_i onto V'_i for each i, such that π induces an isomorphism $\Gamma(X) \to \Gamma'(X^\pi)$ for each $r-2$-clique X of Γ. In particular, if v is any vertex of Γ and v' is its image under π then π induces a cover $\Gamma(v) \to \Gamma'(v')$.

Thus, covers are "local" isomorphisms. For examples see Example 4 in § 3.

Tits [9] showed that each GAB has a <u>universal cover</u> $\tilde{\Gamma} \to \Gamma$, where "universal" refers to the standard type of universal mapping property.

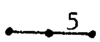
In Examples 1 - 4, the identity map $\Gamma \to \Gamma$ is universal. In Example 4, $\Gamma \to \Gamma(\text{mod } m)$ is a universal cover of $\Gamma(\text{mod } m)$.

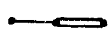
The GAB Γ is <u>simply connected</u> if the identity map $\Gamma \to \Gamma$ is a universal cover. This is closely related to the standard topological notion of simple connectivity. The two notions frequently coincide in practice, for example when $r = 3$.

We can now define <u>rank</u> r <u>buildings</u>. If $r = 2$ this was defined in § 2. Let $r > 2$. Then a <u>building with diagram</u> D is defined recursively to be a GAB Γ with diagram D such that the following hold:

(i) For each vertex v the GAB $\Gamma(v)$ is a building;

(ii) Γ is simply connected;

(iii) D is not  ; and

(iv) If D is  then, for any distinct $v_1, w_1 \in V_1$, $|\Gamma(v_1) \cap \Gamma(w_1) \cap V_2| \le 1$.

The heart of the definition is (i) and (ii). Conditions (iii) and (iv) are a bit irritating. If Γ is finite then (iii) is automatic, by the Feit-Higman theorem

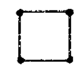
(§ 2). It is conjectured that Example 3 is the only finite GAB with diagram  that fails to satisfy (iv).

<u>The main theorems concerning buildings are the following results of Tits</u>.

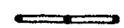
(I) (Tits [7].) Classification of all <u>finite</u> buildings with connected diagram and rank $r \ge 3$.

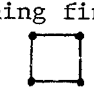
(II) (Tits [7].) Classification of all infinite buildings whose diagrams are those of the crystallographic semiregular polytopes in § 3.

(III) (Tits, unpublished.) Classification of all buildings whose diagrams correspond to semiregular tesselations of $\mathbb{R}^{r-1}$, $r \ge 4$.

Examples of diagrams in (III) are  and . A near-example is , which corresponds to the square lattice in $\mathbb{R}^2$--but of course $r = 3$ here so that (III) does not apply.

(IV) (Tits [9].) Let Γ be a GAB of rank $r \ge 3$. Assume that there is no subdiagram  and that all subdiagrams  correspond to neighborhoods that are buildings. If $\tilde{\Gamma} \to \Gamma$ is a universal cover then $\tilde{\Gamma}$ is a building.

<u>Example</u>. A finite GAB  always has an <u>infinite</u> building  covering it.

Therefore, <u>the study of finite GABs forces the study of infinite ones</u>. For example, Tits' classification theorems give information concerning finite GABs (see [4 ; 5] for the cases of diagrams  and ).

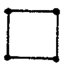
5. Finite examples of GABs, $r > 2$; open problems.

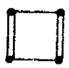
Many finite examples are known, but they have highly limited local behavior: all rank 2 pieces (i.e., bipartite neighborhoods) have small numbers of vertices. Here is a list of some of the corresponding diagrams.

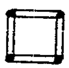

Examples in which the projective planes are PG(2 , 2); examples in which they are PG(2 , 8).


Several classes of examples, related to GF(2), GF(3) or GF(4).

Infinitely many with PG(2 , 2)s and the same universal cover; one with PG(2 , 5)s.

 or 
Infinitely many, using PG(2 , 2)s.


Two examples are known, both using PG(2 , 2)s and the 30 vertex examples in § 2.


30 vertex rank 2 buildings all around.

These are just a sample of the known finite examples of GABs (e.g., [4 ; 5]). All of these, and most of the known explicit examples, are constructed so as to have Aut(Γ) transitive on the set of all maximal cliques of Γ. However, this is really a crutch, and hopefully will not persist. (See [8, pp. 318 - 319] for an indication of how to use infinite groups to construct large numbers of not terribly explicit finite GABs.)

Problems.

(i) Construct more examples, especially without using groups.

(ii) Many known finite GABs with diagram  are related to planar difference sets. There is a good chance

that every planar difference set will produce (via covers) infinitely many finite GABs  [6]. Conceivably, the algebraic approach used to prove the Feit-Higman theorem (via adjacency matrices and eigenvalues) can even be used together with GABs in order to obtain new information concerning difference sets.

(iii) Classify some of the known finite GABs under suitable hypotheses.

(iv) Rank 2 buildings are bipartite of diameter n and girth 2n. Therefore, they are extremal in a natural sense (see [1]). It would be of interest to have extremal problems for r-partite graphs more or less characterizing finite GABs or buildings.

(v) Study further graph theoretic properties of classes of finite GABs. This entire subject is still in its infancy: new properties obtained from new points of view are undoubtedly waiting to be discovered.

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