Grid–symmetric generalized quadrangles*

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Abstract

A generalized quadrangle is classical if it has a grid of axes of symmetry.

In a finite generalized quadrangle $\mathbf{Q}$ of order (s,t) with $s,t>1$, a line L is called an *axis of symmetry* if the group $T(L)$ of all automorphisms ("symmetries") that fix every line meeting L has the maximal possible order s. Moreover, $\mathbf{Q}$ is called *span–symmetric* if there are two disjoint axes of symmetry; we will call $\mathbf{Q}$ *grid–symmetric* if there are two further disjoint axes of symmetry, each of which meets L and M.

Span–symmetric generalized quadrangles were first studied in [Pa] (cf. [PT1]), in view of the known examples $Q(4,q)$ and $Q(5,q)$, arising respectively from quadrics in 4– and 5–dimensional projective spaces. More than 20 years ago it was shown that the generalized quadrangles $Q(4,q)$ are the only span–symmetric ones with $t \neq s^2$ (cf. [Ka, Th1]). While nonclassical examples exist if $t=s^2$, this is not so in the grid–symmetric case:

Theorem. *Any grid–symmetric generalized quadrangle of order* (s,t) *is isomorphic to* $Q(4,s)$ *or* $Q(5,s)$.

Proof. By the result just noted, we may assume that $t=s^2$. There are sets Λ and $\Lambda^\perp$, each consisting of $s+1$ lines of symmetry, where each line in Λ meets each line in $\Lambda^\perp$. Let A and B be the groups generated by the symmetries corresponding to Λ and $\Lambda^\perp$, respectively. By [Th2, 12.5.5], $A \cong B \cong \mathrm{SL}(2,s)$. If $L \in \Lambda$ and $M \in \Lambda^\perp$ then $T(L)$ fixes M and hence normalizes $T(M)$. Also $T(M)$ normalizes $T(L)$, so that these two groups commute since $T(L)\cap T(M)=1$. Thus, A and B are commuting groups each of which is isomorphic to $\mathrm{SL}(2,s)$.

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Let Ω denote the set of points on all lines of Λ, and hence of $\Lambda^\perp$. If x is any point not in Ω then $\Omega \cup x^A$ is the set of points of a $Q(4,s)$–subquadrangle $\mathbf{Q}_x$ [Th2, 12.5.5]. If $M \in \Lambda^\perp$ then $T(M)$ fixes each line of $\mathbf{Q}_x$ meeting M and hence acts on the union $\mathbf{Q}_x$ of these lines. Thus, AB acts on $\mathbf{Q}_x$, and hence acts in the natural manner as $\Omega^+(4,s)$ on the space $\mathbf{P}_x = \mathrm{PG}(4,s)$ underlying $\mathbf{Q}_x$, fixing the point m of $\mathbf{P}_x \backslash \mathbf{Q}_x$ perpendicular to $\langle \Omega \rangle$. Note that $AB \cong \Omega^+(4,s)$: if s is odd and z_A and z_B are the involutions in A and B, respectively, then $z_A z_B = 1$ on $\mathbf{Q}_x$ for each point $x \notin \Omega$, and hence is 1 on $\mathbf{Q}$.

Note that, if $x \notin \Omega$ as above, then $(AB)_x \cong \mathrm{PSL}(2,s)$. For, x lies on the line of $\mathbf{P}_x$ joining m and some point n of $\langle \Omega \rangle \backslash \Omega$, so that the stabilizer $(AB)_x$ fixes n. However, $(AB)_n \cong \Omega(3,s) \cong \mathrm{PSL}(2,s)$ has no proper subgroup of index $(2, s-1)$. Since $(AB)_n$ permutes the $(2, s-1)$ points of $\mathbf{Q}_x$ on the line $\langle m, n \rangle$, it follows that $(AB)_x = (AB)_n \cong \mathrm{PSL}(2,s)$.

Now consider any point y of $\mathbf{Q}$ not in $\Omega \cup x^A$ and the resulting point-orbit y^A and subquadrangle. As in the preceding paragraph, $G := (AB)_y \cong \mathrm{PSL}(2,s)$. Here G acts on $\mathcal{O} := y^\perp \cap \mathbf{Q}_x$, which is an ovoid of $\mathbf{Q}_x$ [PT2, p. 26]: each of the s^2+1 lines through y meets $\mathbf{Q}_x$, and no two of the resulting s^2+1 points are perpendicular.

Under the Klein correspondence for a suitable quadric of $\mathbf{P} = \mathrm{PG}(5,q)$ containing $\mathbf{Q}_x$, the ovoid $\mathcal{O}$ produces a spread of lines in $\mathrm{PG}(3,s)$ and hence also a translation plane π of order s^2, with kernel containing $\mathrm{GF}(s)$. Moreover, under this correspondence, the group $AB \cong \Omega^+(4,s)$ produces a subgroup of $\mathrm{GL}(4,s)$, isomorphic to $A \times B$, that has a subgroup $\hat{G} \cong \mathrm{PSL}(2,s)$ or $\mathrm{SL}(2,s)$ produced by G; moreover $\hat{G}$ preserves the spread. If q is odd then $\hat{G} \not\cong \mathrm{PSL}(2,s)$ since all involutions in $A \times B$ lie in its center. For all q it follows that G produces a collineation group $\hat{G} \cong \mathrm{SL}(2,s)$ of π.

All translation planes having the preceding properties are known [Sch, Wa]: the nondesarguesian ones are Hall, Hering, Walker and Ott-Schaeffer planes. It is easy to check that, for each of these nondesarguesian planes, the corresponding ovoid spans $\mathbf{P}$, whereas our ovoid $\mathcal{O}$ lies in $\mathbf{Q}_x$ and hence in the hyperplane $\mathbf{P}_x$ of $\mathbf{P}$. Hence π is desarguesian and $\mathcal{O}$ is an elliptic quadric.

Thus, $y^\perp \cap \mathbf{Q}_x$ is an elliptic quadric of $\mathbf{Q}_x$ for each point y of $\mathbf{Q} \backslash \mathbf{Q}_x$. Consequently, our original generalized quadrangle is classical [TP, Br]. ■

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