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SOME HIGHLY GEOMETRIC LATTICES

Riassunto. — Vengono esaminati parecchi argomenti riguardanti reticoli geometrici finiti che soddisfano condizioni geometriche addizionali. Si esaminano spazi localmente proiettivi e variazioni al teorema di Dembowski–Wagner.

Vengono descritti nei particolari teoremi riguardanti le tecniche di immersione, e poi applicati a certi sistemi di Steiner. Infine, viene presentato un fatto di teoria dei gruppi relativo a reticoli geometrici altamente simmetrici.

1. Introduction

This paper is dedicated to the memory of Peter Dembowski.

In 1960, Dembowski and Wagner [7] proved their well-known characterization of finite projective spaces in terms of designs. In 1967, a generalization of their theorem was proved, simultaneously characterizing finite projective and affine spaces [13]. Last year, I learned that these theorems could be regarded as results concerning finite geometric lattices. This then led to a search for more general results. Theorems have been obtained concerning the embedding of some very geometric types of geometric lattices into projective spaces; these will be described here. Finite geometric lattices having highly transitive automorphism groups will also be discussed.

2. Embedding theorems

Throughout our discussion, lattices will always be finite, although most of the results of this section have infinite analogues.

Let G be a geometric lattice. Each element $X \in G$ has a dimension dim X, with $\dim 0 = -1$ and $\dim 1 = \dim G$ the dimension of G. (This is the concept of dimension used in projective geometry.) Each $X \in G$ is a join of points (i.e., atoms). If $X, Y \in G$, $\dim X + \dim Y \geq \dim X \vee Y + \dim X \wedge Y$. Thus, if p is a point and $X \in G$, then $\dim (X \vee p) - \dim X = 0$ or 1.

If $W \in G$, $[W, 1]$ denotes the interval $\{X \in G \mid X \geq W\}$, which is a geometric lattice of dimension $\dim G - \dim W - 1$.

If G and H are geometric lattices, an *isometry* from G to H is an order-preserving injective mapping $\theta : G \rightarrow H$ such that $1^\theta = 1$, $\dim X^\theta = \dim X$

if $X \neq 1$ or $\dim X \leq 1$, and $(X \vee Y)^\theta = X^\theta \vee Y^\theta$ whenever $X, Y \in G$ satisfy $X \vee Y \neq 1$. For example, the inclusion mapping is an isometry from the lattice consisting of 0, 1, and the points, lines, and planes of AG (d, q), into PG (d, q).

In [13], the following situation was, in effect, studied. Let G be a geometric lattice of dimension $n \geq 3$. Assume that each hyperplane (element of dimension $n - 1$) has k points, and any intersection of two different hyperplanes has μ or 0 points, where k and μ are constants satisfying $k > \mu > 0$. A non-degeneracy condition was also assumed. It was then proved that G is PG(n, q) or AG(n, q) for some q, provided $n \geq 4$. This is false for $n = 3$; however, any of a variety of natural additional conditions sufficed to characterize PG$(3, q)$ and AG$(3, q)$. In particular, if any two different hyperplanes have exactly μ common points, G is PG(n, q); this is precisely the dual of the Dembowski-Wagner Theorem.

When $n = 3$, the lines and planes through a point form a projective plane of order q, where q does not depend on the point. If G is neither PG$(3, q)$ nor AG$(3, q)$, further numerical information was also obtained in [13]. Using the same information, Doyen and Hubaut [8] were able to obtain the following two possibilities for μ, q, k, and the number v of points:

(I) $$q = \mu^2 \quad , \quad k = \mu(\mu^2 - \mu + 1) \quad , \quad v = k + q^2(\mu - 1) \, ,$$

and

(II) $$q = \mu^3 + \mu \quad , \quad k = \mu^2(\mu^2 - \mu + 1) \quad , \quad v = k + q^2(\mu - 1).$$

The only case known to occur is the lattice W_{22} associated with the Mathieu group M_{22}, which corresponds to case (I) with $\mu = 2$, $q = 4$, $k = 6$, and $v = 22$.

The question now arises: how crucial were the numerical assumptions made in [7] and [13]? Thus, suppose G is an n-dimensional geometric lattice such that $\dim X \wedge Y = n - 2$ or -1 for any two different hyperplanes X and Y. If $n \geq 4$, it seems likely that there is an isometry from G into an n-dimensional modular geometric lattice. If $n \geq 3$, we conjecture that the same is true if $\dim X \wedge Y = n - 2$ for any two different hyperplanes X and Y. (This is, however, definitely not true for infinite geometric lattices of any dimension $n \geq 3$.) So far, the following is the closest we have come to such a result.

THEOREM 1. *Let* G *be an n-dimensional geometric lattice. Suppose* $1 \leq e \leq f \leq n - 1$, $e \neq n - 1$, *and* $e + f \geq n$. *Then there are* $E, F \in G$ *with* $\dim E = e$, $\dim F = f$, $E \vee F = 1$, *and* $\dim E \wedge F = e + f - n - 1$, *unless the following hold.*

(SE) There is an isometry $\theta : G \to M$, with M an n-dimensional modular geometric lattice, satisfying the following conditions for all $W \in G$.

(i) If $\dim W \leq n-4$, each element of $[W^\theta, 1_M]$ of dimension $\leq \dim W + 2$ is the intersection (in M) of those members of G^θ containing it; and

(ii) If $\dim W = n-3$, $[W, 1_G]^\theta$ is either $[W^\theta, 1_M]$ or is obtained from $[W^\theta, 1_M]$ by removing a line and all its points.

In general, a geometric lattice G of dimension $n \geq 3$ will be said to be *strongly embedded* in the modular geometric lattice M if $\dim M = n$ and there is an isometry $\theta : G \to M$ satisfying (SE). Note that condition (ii) permits $[W, 1]^\theta$ to be a (possibly degenerate) projective plane or an affine plane.

It is straightforward to show that E and F exist in Theorem 1 unless $[W, 1]$ is modular for each $W \in G$ of dimension $n-4$. Therefore, Theorem 1 is a special case of the next result (see [18]):

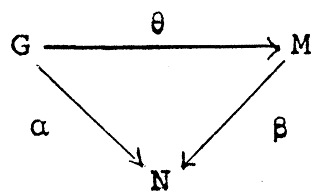
THEOREM 2. *Let* G *be a geometric lattice of dimension* $n \geq 4$. *Suppose that, for each* $W \in G$ *of dimension* $n-4$, $[W, 1]$ *can be strongly embedded in a modular geometric lattice. Then* G *can be strongly embedded in a modular geometric lattice.*

Thus, if each such $[W, 1]$ is a large chunk of a 3-dimensional modular geometric lattice, the same is true of G. The example $G = W_{22}$ shows that the analogues of Theorems 1 and 2 are false when $\dim G = 3$.

Before describing applications in § 3, we turn to what is almost, but not quite, a more general result. Note that the modular geometric lattices considered thus far need not be projective spaces. If we specialize them to projective spaces, Theorem 2 is contained in Theorem 3.

Some more definitions are required. Fix a finite field K. Projective K-spaces will be assumed to be finite-dimensional.

A K-*envelope* of a geometric lattice G is a pair (M, θ) consisting of a projective K-space M and an isometry $\theta : G \to M$, such that, whenever $\alpha : G \to N$ is an isometry from G into a projective K-space N such that $\bigvee (G^\alpha - \{1_N\}) = 1_N$, there is a *unique* isometry $\beta : M \to N$ making the following a commutative diagram.



If M is a projective K-space, and if G is a geometric lattice strongly embedded in M via the isometry $\theta : G \to M$, then (M, θ) is usually a K-envelope of G. However, even in this situation it is essential that the field be specified if one wants a universal mapping property; for example, there is an isometry $\theta : AG(2, 3) \to PG(2, 4)$ such that $(PG(2, 4), \theta)$ is a GF(4)-envelope of AG(2, 3).

Note that, if (M, θ) is a K-envelope of G, dim M and dim G may be different. For example, this is the case when G consists of $0, 1$, and the points, lines, and planes of a projective K-space M, where θ is the inclusion map.

One last definition: a geometric lattice G is K-*rigid* if, for each isometry $\varphi : G \to N$ with N a projective K-space and $\vee (G^{\varphi} - \{1_N\}) = 1_N$, the identity is the only collineation of N inducing the identity on G^{φ}. For example, if G is 1-dimensional and has $|K| + 1$ or $|K|$ points, it is trivially K-rigid; however, if $|K| > 4$, G has no K-envelope.

On the other hand, G is K-rigid if it has a K-envelope. To see this, let (M, θ) be a K-envelope of G, $\varphi : G \to N$ an isometry from G into a projective K-space N such that $\vee (G^{\varphi} - \{1\}) = 1$, and σ a collineation of N inducing the identity on G^{φ}. There is a unique isometry $\beta : M \to N$ such that $\theta\beta = \varphi$. Then $\vee (M^{\beta} - \{1\}) = 1$ (since $M^{\beta} - \{1\} \supseteq G^{\theta\beta} - \{1\} = G^{\beta} - \{1\}$). Hence, $M^{\beta} = N$. By hypothesis, $\varphi\sigma = \varphi$. Consequently, $\varphi = \varphi\sigma = \theta(\beta\sigma)$, where $\beta\sigma : M \to N$. The uniqueness of β yields $\beta = \beta\sigma$, so σ is the identity on $M^{\beta} = N$, as required.

The following result is proved in [19].

THEOREM 3. *Let* K *be a finite field, j a positive integer, and* G *a geometric lattice of dimension* $\geq j + 3$. *Suppose that, for all* $W \in G$, $[W, 1]$ *has a* K-*envelope whenever* $\dim [W, 1] = j + 1$ *or* $j + 2$, *while* $[W, 1]$ *is* K-*rigid whenever* $\dim [W, 1] = j$. *Then* G *has a* K-*envelope.*

Once again, this states in effect that, if enough intervals $[W, 1]$ are large chunks of projective K-spaces, the same will be true for G. The proofs of Theorems 2 and 3 are similar. The idea in either case is as follows. By induction, for each point p we may assume that $[p, 1]$ satisfies the conclusions of the theorem. Glue the corresponding modular lattice onto p, making sure that this is done in a coherent manner. The result is a poset H containing G such that each $[p, 1_H]$ is a modular geometric lattice. In [18] it was proved that such a poset H has a canonical embedding into a modular geometric lattice.

3. APPLICATIONS

Theorem 3 implies Tutte's well-known characterization of binary geometric lattices (see [2]). The proof merely involves a straightforward check in the cases of dimension ≤ 3, after which Theorem 3 applies with $K = GF(2)$ and $j = 1$. However, Theorem 3 has limited applicability to such problems over other fields. This is due to the strength of the uniqueness part of the definition of K-envelopes. For example, if $|K| > 2$ a triangle does not have a K-envelope (in fact, it is not even K-rigid).

In § 2 we discussed some geometric lattices G in which $[p, 1]$ is a projective plane for each point p. Suppose G is a geometric lattice of dimension $n \geq 4$ in which $[p, 1]$ is a projective space for each point p. By Theorem 2 or 3, G can be regarded as a sublattice of a projective space $M = PG(n, q)$ such that the inclusion map $G \to M$ is an isometry. It is easy to see that each such G can be obtained from M by the following natural generalization of the construction of AG (n, q) from PG (n, q). Take any family of hyperplanes of M, and delete them and all their subspaces. The result is a lattice G, which will have the desired properties if and only if each of its elements is a join of points of G.

Similarly, Theorems 2 and 3 apply to geometric lattices G of dimension ≥ 4 such that each $[p, 1]$ is an affine space. However, this time it seems much harder to specify in terms of projective spaces exactly what lattices occur. In view of the interest in t-designs, it is natural to consider the following similar situation.

THEOREM 4. *Let* D *be a t-design with* $t \geq 3$. *Assume that, for each set* W *of* $t - 2$ *points, the points not in* W *and the blocks containing* W *form the design of points and hyperplanes of a finite affine space. Then one of the following holds.*

(i) $t = k = v - 2$, *so* D *is degenerate.*

(ii) $t = 3$ *and* D *is an inversive plane.*

(iii) $t = 4$ *and* D *is the* $4 - (11, 5, 1)$ *design associated with the Mathieu group* M_{11}.

(iv) $t = 5$ *and* D *is the* $5 - (12, 6, 1)$ *design associated with the Mathieu group* M_{12}.

(v) $t = 4$, D *is a* $4 - (171, 15, 1)$ *design, and there is a point* p *such that* D_p *is a non-miquelian inversive plane of order 13. (No such inversive plane is known.)*

Proof (see [18], § 5). Let G be the geometric lattice consisting of the sets of points having $\leq t - 1$ elements, the blocks of D, and 1. Then $|W| = t - 2$ means $\dim W = t - 3$. By Dembowski [5], $[W, 1]$ is an affine plane of order q, say, where q does not depend on W. Hence, (ii) holds if $t = 3$.

Suppose $t = 4$. An easy count shows that there are

$$(q^2 + 2)(q^2 + 1) q^2 (q^2 - 1)/(q + 2)(q + 1) q (q - 1)$$

3-spaces. If p is a point, $[p, 1]$ is an inversive plane of order q. Thus, if q is even then q is power of 2 by the fundamental result of Dembowski [4]. It follows that $q = 2, 3, 4, 8$ or 13. Here, $q = 2$ and 3 correspond to (i) and (iv) (see [24] and [25]).

Suppose $q = 4, 8$ or 13. If $q = 4$ or 8, each $[p, 1]$ is egglike (by Dembowski's result [4]); when $q = 13$, assume the same is true, so (v) does not hold by a result of Barlotti and Panella ([6], p. 49). Then $[p, 1]$ has a GF (q)-envelope. By Theorem 2, or Theorem 3 with $j = 1$, G has a GF (q)-envelope (M, θ), where $M = PG(4, q)$. We may thus assume that $G \subset M$ and θ is the inclusion map.

Since planes of G have just 3 points, skew lines of G are skew in M. Thus, the total number of points of M on all lines of G is $(q^2+2)+\binom{q^2+2}{2}(q-1)>(q^5-1)/(q-1)$, which is ridiculous.

Thus, $t=4$ implies that (iii) or (v) holds. Finally, if $t\geq 5$, (iv) follows from [25] and another count.

The main part of the above proof consisted of the classification of $t-(q^2+t-2, q+t-2, 1)$ designs with $t\geq 4$, except for the possibility (v). (The case $t=q=4$ is due to Witt [25].) The structure of our proof is, perhaps, interesting. First, Dembowski's theorem is applied. Then, instead of counting as is usual with designs, or using properties of egglike inversive planes or desarguesian affine planes, we invoked a general lattice-theoretic result. Only now is an easy counting argument used to obtain the desired contradiction.

Consider the cases (iii) and (iv) again. By Theorem 2 or 3, G has a GF(3)-envelope (M, θ), where $M = PG(t, 3)$. The definition of K-envelopes implies that every automorphism of G extends to a unique collineation of M. This provides a non-computational proof of the existence of the well-known 6-dimensional projective GF(3)-representation of M_{12} (compare [1]).

4. The Dembowski-Wagner Theorem

Using geometric lattices, the Dembowski-Wagner Theorem can be generalized as follows.

THEOREM 5. *A finite incidence structure* D *of points and blocks is isomorphic to the design of points and hyperplanes of a finite projective space if and only if the following conditions hold.*

(i) *Each pair of points is on the same number* λ *of blocks.*

(ii) *Each block is on at least* 3 *points; there is at least one block.*

(iii) *Each point is on at least* $\lambda+1$ *blocks, and some point is on at least* $\lambda+2$ *blocks.*

(iv) *For distinct points* x *and* y, *define the line joining them by* $xy=\cap\{B \mid B$ *is a block on* $x, y\}$. *Then each line meets each block.*

Proof. Necessity is clear. Assume (i)–(iv). A standard incidence matrix argument (see [6], p. 20) shows that, by (i) and (iii), D has at least as many blocks as points. Now pass to the dual $\tilde{D}$ of D. Then, if B and C are different blocks and $x \notin B\cap C$, there is a unique block containing $B\cap C$ and x: existence is (iv), while uniqueness follows from (i). The set of intersections of blocks thus forms a geometric lattice G (see [2]). By [9], $\tilde{D}$ has at least as many blocks as points, with equality if and only if G is modular. By (i) and (iii), G must be a projective space.

We have not been able to generalize the main result of [13] in a similar manner, although such a generalization very likely exists.

5. The deficiency of a geometric lattice

In [3], Dembowski proved what amounts to the following result. Let G be a 2-dimensional geometric lattice, and suppose each line has k or $k+1$ points. Then G is either a projective plane of order $k-1$, or can be extended to a projective plane of order k. More general results due to Dembowski are presented in [6], § 7.4. This result can be generalized in a different direction, as follows.

Let G be a geometric lattice of dimension $n \geq 2$. Let $m_1(G)$ be the maximal number of hyperplanes per element of dimension $n-2$, and $m_0(G)$ the minimal number of points per line. Then $m_1(G) - m_0(G)$ is a non-negative integer, the *deficiency* of G. If the deficiency is 0, G is a Boolean algebra or a projective space.

Theorem 6. *Let* G *be a geometric lattice of dimension* $n \geq 2$ *and deficiency* 1. *Then there is an isometry from* G *into an* n*-dimensional projective space having* $m_1(G)$ *points per line.*

Although this contains Tutte's characterization of binary geometric lattices (when $m_1(G) = 3$), the proof we have for the case $m_1(G) \geq 4$ does not work if $m_1(G) = 3$. When $n = 2$, Theorem 6 is just Dembowski's result. For $n \geq 3$, the unpublished proof requires Dembowski's result together with classical geometric arguments.

6. Automorphism groups

In 1871, Jordan [12] initiated the study of permutation groups Γ satisfying the following conditions. Γ is 2-transitive on a finite set S, but not k-transitive on S, where $2 < k < |S| - 1$; and there is a set B if k points such that the pointwise stabilizer $\Gamma(B) = \{\gamma \in \Gamma \mid x^\gamma = x \mid \forall x \in B\}$ of B is transitive on S — B. During the next half century, Jordan and others proved results concerning such groups Γ of a group theoretic or combinatorial nature. There has recently been a resurgence of interest in this situation ([10], [11], [13], [14], [15], [16], [17], [21], [22]).

In [14], the following facts were proved about Γ. Let G consist of all the subsets of S which are intersections of subsets of $\{B^\gamma \mid \gamma \in \Gamma\}$ (where $B^\gamma = \{x^\gamma \mid x \in B\}$). Then G is a geometric lattice (this is the content of [14], (3.10), although geometric lattices were not explicitly mentioned there). Each hyperplane has more than $n = \dim G$ points. If $X \in G$, then $\Gamma(X)$ is transitive on S — X. Γ is transitive on $\{X \in G \mid \dim X = i\}$ whenever $0 \leq i \leq n$. From this it follows easily that $|S| \geq 2k$, a result due to Marggraff [20].

The only known possibilities for G are (up to isomorphism)

(i) 1 and the set of all subspaces of dimension $\leq n-1$ of a finite projective or affine space of dimension $\geq n$, and

(ii) one of the lattices W_{22}, W_{23} and W_{24} associated with the Mathieu groups M_{22}, M_{23} and M_{24} (compare [24], [25], and [14]).

According to [14], if $|S| \leq 6k$ then G must be one of these types of lattices.

The following is another (but unpublished) result of the same type.

THEOREM 6. *Let* G *be a geometric lattice of dimension* $n \geq 2$, *in which each hyperplane has more than* n *points. Let* Γ *be an automorphism group of* G *2-transitive on the set* S *of points and transitive on the set of hyperplanes of* G. *Assume further that the pointwise stabilizer of a hyperplane* B *is transitive on the set* S — B *of points not in* B. *If* $|S - B|$ *is a prime power, then* G *is isomorphic to a projective space, an affine space over* GF (2), W_{22}, W_{23}, *or* W_{24}.

Alternatively, the same conclusion holds if $\Gamma(B)$ has an abelian subgroup transitive on S — B. Special cases of Theorem 6 are found in [14], [15], [16], [21] and [22].

A result of a somewhat different type concerning automorphism groups of geometric lattices is proved in [18]. All of these results indicate that geometric lattices having sufficiently highly transitive automorphism groups are of known type.

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