Math. Z. 124, 315–318 (1972)

On Incidence Matrices of Finite Projective and Affine Spaces

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It is well-known that the rank of each incidence matrix of all points *vs.* all e-spaces of a finite d-dimensional projective or affine space is the number of points of the geometry, where $1 \leqq e \leqq d-1$ (see [1], p. 20). In this note we shall generalize this fact:

Theorem. *Let $0 \leqq e < f \leqq d-e-1$, and let $M_{e,f}$ be an incidence matrix of all e-spaces vs. all f-spaces of $PG(d,q)$ or $AG(d,q)$. Then the rank of $M_{e,f}$ is the number of e-spaces of the geometry.*

We shall only prove the theorem in the case of $AG(d,q)$. The projective case is similar and simpler. We note that the same proof shows that, *if $1 \leqq e < f \leqq d-e$, an incidence matrix of all e-sets vs. all f-sets of a set of d points has rank $\binom{d}{e}$.*

The relevant definitions are found in [1], §§ 1.3 and 1.4. The dimension of a subspace X of a projective space will be denoted $\dim(X)$. The empty subspace has dimension -1.

Proof. Let E and F denote any e-space and f-space, respectively. Set

$$(E,F)=\begin{cases}1 & \text{if } E\subset F\\ 0 & \text{if } E\not\subset F.\end{cases}$$

For a suitable ordering of e-spaces and f-spaces we have $M_{e,f}=((E,F))$. Let $R(E)$ denote the row of $M_{e,f}$ corresponding to the e-space E.

Assume that there is a nontrivial dependence relation among the rows of $M_{e,f}$. This may be assumed to have the form

$$R(E^*)=\sum_{E\neq E^*} a(E)\,R(E) \tag{1}$$

for some e-space E^*, where each $a(E)$ is a real number. Let Γ be the group of all collineations of $AG(d,q)$ taking E^* to itself. If $\alpha\in\Gamma$ then $(E^\alpha, F^\alpha)=(E,F)$.

Then (1) implies that, for all F,

$$(E^*, F) = (E^*, F^\alpha) = \sum_E a(E^\alpha)(E^\alpha, F^\alpha)$$
$$= \sum_E a(E^\alpha)(E, F),$$

so that $R(E^*) = \sum_{E \neq E^*} a(E^\alpha) R(E)$. Thus,

$$\begin{aligned} |\Gamma|\, R(E^*) &= \sum_\Gamma \sum_{E \neq E^*} a(E^\alpha)\, R(E) \\ &= \sum_{E \neq E^*} R(E) \sum_\Gamma a(E^\alpha). \end{aligned} \tag{2}$$

Let $\mathfrak{S}$ be the set of all ordered pairs (i, j) of integers satisfying the conditions $-1 \leqq i, j \leqq e-1$ and $j = i$ or $i+1$. Order $\mathfrak{S}$ lexicographically: $(i_1, j_1) < (i_2, j_2)$ if either $i_1 < i_2$ or $i_1 = i_2$ and $j_1 < j_2$.

We now embed $AG(d, q)$ in $PG(d, q)$. Let H_∞ be the hyperplane at infinity. If $(i, j) \in \mathfrak{S}$, let $\mathscr{E}(i, j)$ be the set of e-spaces $E \not\subset H_\infty$ such that $\dim(E \cap E^* \cap H_\infty) = i$ and $\dim(E \cap E^*) = j$. The $\mathscr{E}(i, j)$ are precisely the orbits of Γ of e-spaces other than E^*. Fix $E_{ij} \in \mathscr{E}(i, j)$. Then (2) implies that

$$\begin{aligned} |\Gamma|\, R(E^*) &= \sum_{\mathfrak{S}} \sum_{\mathscr{E}(i,j)} R(E) \sum_\Gamma a(E^\alpha) \\ &= \sum_{\mathfrak{S}} \sum_{\mathscr{E}(i,j)} R(E)\, |\mathscr{E}(i,j)| \sum_\Gamma a(E_{ij}^\alpha) \\ &= \sum_{\mathfrak{S}} \Big[|\mathscr{E}(i,j)| \sum_\Gamma a(E_{ij}^\alpha)\Big] \sum_{\mathscr{E}(i,j)} R(E). \end{aligned}$$

There is thus a dependence relation of the form

$$R(E^*) = \sum_{\mathfrak{S}} b_{ij} \sum_{\mathscr{E}(i,j)} R(E) \tag{3}$$

with b_{ij} real.

Let $(m, n) \in \mathfrak{S}$. Since $e + f \leqq d - 1$ there is an f-space F_{mn} such that

$$\dim(F_{mn} \cap E^* \cap H_\infty) = m \quad \text{and} \quad \dim(F_{mn} \cap E^*) = n.$$

As $n \leqq e-1$, $F_{mn} \not\supset E^*$. By (3),

$$0 = \sum_{\mathfrak{S}} b_{ij} \sum_{\mathscr{E}(i,j)} (E, F_{mn}) \tag{4}$$

whenever $(m, n) \in \mathfrak{S}$.

Let (i, j), $(m, n) \in \mathfrak{S}$. If $(E, F_{mn}) = 1$ for some $E \in \mathscr{E}(i, j)$ then $E \subset F_{mn}$ implies that $i \leqq m, j \leqq n$. In particular, if $(i, j) > (m, n)$ then

$$\sum_{\mathscr{E}(i,j)} (E, F_{mn}) = 0. \tag{5}$$

Also, there is an e-space $E \subset F_{ij}$ such that $E \cap E^* \cap H_\infty = F_{ij} \cap E^* \cap H_\infty$ and $E \cap E^* = F_{ij} \cap E^*$. Thus,

$$\sum_{\mathscr{E}(i,j)} (E, F_{ij}) \neq 0. \tag{6}$$

By (4) and (5),

$$0 = \sum_{(i,j) \leqq (m,n)} b_{ij} \sum_{\mathscr{E}(i,j)} (E, F_{mn}) \tag{7}$$

for all $(m, n) \in \mathfrak{S}$. This is a system of $|\mathfrak{S}|$ equations in the $|\mathfrak{S}|$ unknowns b_{ij}, $(i,j) \in \mathfrak{S}$. If we use the ordering of $\mathfrak{S}$, the coefficient matrix is triangular, with nonzero diagonal entries by (6). It follows that $b_{ij} = 0$ for all $(i,j) \in \mathfrak{S}$, contradicting (3). This completes the proof.

We note incidentally that the matrices $M_{e,f}$ have the following property: if

$$0 \leqq e < f < g \leqq d-1,$$

then $M_{e,f} M_{f,g} = c_{e,f,g} M_{e,g}$, where $c_{e,f,g}$ is the number of f-spaces contained in a g-space G and containing an e-space contained in G.

Corollary 1. *If* $0 \leqq e \leqq d-e-1$, *then a collineation of* $PG(d,q)$ *induces similar permutations on the sets of e-spaces and* $(d-e-1)$*-spaces.*

Proof. [1], p. 22.

Corollary 2. *Let* Γ *be a collineation group of* $PG(d,q)$ *or* $AG(d,q)$. *If*

$$0 \leqq e < f \leqq d-e-1,$$

then Γ *has at least as many orbits of f-spaces as e-spaces. Moreover, these quantities are equal in the projective case if* $f = d-e-1$.

Proof. [1], pp. 20–22.

Corollary 3. *Let* Γ *be a collineation group of* $PG(d,q)$ *or* $AG(d,q)$ *transitive on f-spaces. If* $0 \leqq e < f \leqq d-e-1$ *then* Γ *is transitive on e-spaces, and its rank on f-spaces is at least as large as that on e-spaces. Moreover, these ranks are equal in the projective case if* $f = d-e-1$.

Proof. See the proof of [2], Theorem 4.4.

We remark that, by using [1], p. 21, together with the analogue of our theorem for incidence matrices of subsets of a finite set, we also obtain an elementary proof of Theorem 1 of Livingstone and Wagner [3].

Parts of these corollaries are due to Wagner [5] in the projective case.

Added in Proofs. The corollaries can also be easily deduced from the following facts. Let θ_e be the permutation character obtained from the action of the full collineation group of $PG(d,q)$ or $AG(d,q)$ on the e-spaces of the geometry, where $0 \leqq e \leqq d-1$. If $0 \leqq e \leqq d/2$, then $\theta_{e-1} \subset \theta_e = \theta_{d-1-e}$ for $PG(d,q)$, and $\theta_{e-1} \subset \theta_{d-e} \subset \theta_e$ for $AG(d,q)$. These inclusions can be proved by calculating (θ_e, θ_f) whenever $0 \leqq e \leqq f \leqq d-1$. In the projective case, this result and this calculation are essentially contained in [4], pp. 276–278.

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(Received September 30, 1970)