

SEMIFIELDS ARISING FROM IRREDUCIBLE SEMILINEAR TRANSFORMATIONS

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Abstract

A construction of finite semifield planes of order n using irreducible semilinear transformations on a finite vector space of size n is shown to produce fewer than $\sqrt{n} \log_2 n$ different nondesarguesian planes.

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1. Introduction

Let $V = V_K$ be a d -dimensional vector space over a finite field K . Suppose that $T \in \Gamma L(V_K)$ is an *irreducible semilinear transformation*: 0 and V are the only T -invariant subspaces of V . (The simplest example is $V = K$ and $T \in \text{Aut}(K)$.) Then $\sum_0^{d-1} T^i K$ is a presemifield [5], so that there is a corresponding semifield plane π_T (see Section 2 below). While it appears that there might be quite a few projective planes obtained in this manner, the purpose of this paper is to show that this is not the case.

THEOREM 1.1. *Fewer than $\sqrt{n} \log_2 n$ pairwise nonisomorphic nondesarguesian semifield planes π_T of order n are obtained from irreducible semilinear transformations T on vector spaces of size n .*

A weaker bound announced in [6] highlighted remarks concerning the relatively small number of known semifield planes. Many standard results concerning linear transformations have been generalized to the semilinear case [4, 2], but these do not appear to give the desired information concerning irreducible transformations. In Section 3 we develop enough machinery concerning semilinear transformations to deduce the theorem.

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2. Semifield planes

A finite *presemifield* is a finite vector space V together with a product $a * b$ that is left and right distributive and satisfies the condition that $a * b = 0$ implies $a = 0$ or $b = 0$. This produces an affine plane (a *semifield plane* [1, Section 5.3]) with point set V^2 and lines $x = c$ and $y = m * x + b$. There is a simple, elegant construction of finite presemifields due to Jha and Johnson [5], using an irreducible semilinear transformation T on a d -dimensional vector space V over a finite field K . Namely, the set $\mathcal{S}_T := \sum_0^{d-1} T^i K$ consists of $|V|$ additive maps $V \rightarrow V$, with all nonzero maps invertible; define $a * b = f(a)(b)$, $a, b \in V$, for an arbitrary additive isomorphism $f: V \rightarrow \mathcal{S}_T$. This produces a presemifield and hence also an affine plane π_T . Different choices for f produce isomorphic planes π_T [1, p. 135].

We repeat the elementary proof in [6] that, if at least one of the $k_i \in K$ is not zero, then the element $\sum_0^{d-1} T^i k_i$ of \mathcal{S}_T is invertible. If this transformation is not invertible then there is some nonzero vector v such that $\sum_0^{d-1} T^i (k_i v) = 0$. Then there is some j such that $1 \leq j \leq d$ and $0 \neq T^j(k_j v) = -\sum_0^{j-1} T^i(k_i v)$. Since $TK = KT$, we have $T(KT^{j-1}(v)) = KT(T^{j-1}(k_j v)) \subseteq \sum_0^{j-1} KT^i(v)$, so that the latter is a proper T -invariant subspace, whereas T is irreducible.

If T is a linear transformation then this construction produces a field in the standard manner. In general, unlike in the case of fields, if T and T' generate the same cyclic group then the planes π_T and $\pi_{T'}$ might not be isomorphic since \mathcal{S}_T is not T -invariant.

However, $\Gamma L(V)$ -conjugates of T produce $\Gamma L(V)$ -conjugate sets \mathcal{S}_T and hence isomorphic planes π_T (but not conversely, as is easily seen using $\text{GF}(|V|)$). Therefore, in the next section we focus on conjugacy of irreducible semilinear transformations.

3. Proof of Theorem 1.1

We begin with the following result.

PROPOSITION 3.1. *Let T be an irreducible σ -semilinear transformation on a finite vector space V over a finite field K . Then there is a decomposition*

$$V = V_1 \oplus \cdots \oplus V_t \tag{3.2}$$

of V into subspaces V_i permuted cyclically by T such that $T^t|_{V_1}$ is a 1-dimensional semilinear map over an extension field of K . Moreover, t divides the order of σ , and the map $T^t|_{V_1}$ uniquely determines T up to $\text{GL}(V)$ -conjugacy.

PROOF. We will proceed in several steps. Throughout the proof, V will always denote a vector space over K . Whenever a subspace of V is viewed as a vector space over another field, or the field involved needs to be emphasized, we will add that field as a subscript.

STEP 1. Let s be the order of σ and $E := C_K(\sigma)$. Clearly V is a vector space over E and $T^s \in \text{GL}(V) \leq \text{GL}(V_E)$. Let $\mu(x) \in E[x]$ be the minimal polynomial of T^s on V_E . We claim that $\mu(x)$ is irreducible. For, if $g(x) \in E[x]$ is a proper nontrivial

divisor of $\mu(x)$, then $\text{Ker } g(T^s)$ is a proper nontrivial subspace of V_E . Since both K and T commute with $g(T^s)$, they leave invariant the K -space $\text{Ker } g(T^s)$, contrary to the irreducibility of T on V .

STEP 2. Let $\mu_1(x) \in K[x]$ be an irreducible factor of $\mu(x)$. Then for some $t \mid s$, $\mu(x) = \prod_{i=1}^t \mu_i(x)$ where the polynomials $\mu_i(x) := \mu_1^{\sigma^{i-1}}(x) \in K[x]$ are distinct and irreducible. For $1 \leq i \leq t$, let $V_i := \text{Ker}(\mu_i(T^s))$. Then (3.2) holds with $T(V_i) = V_{i+1}$ (the subscripts are mod t), and T^s has minimal polynomial $\mu_i(x)$ on $(V_i)_K$. Moreover, $m_1 := T^s|_{V_1}$ is K -linear with irreducible minimal polynomial $\mu_1(x) \in K[x]$, so that $L := K[m_1]$ is a subfield of $\text{End}(V_1)$ and V_1 is a vector space over L .

We always let v_1 denote an arbitrary nonzero vector of V_1 . We have

$$T^t(km_1v_1) = k^{\sigma^t} T^t(T^s(v_1)) = k^{\sigma^t} T^s(T^t(v_1)) = k^{\sigma^t} m_1 T^t(v_1)$$

for $k \in K$. It follows that $T^t|_{V_1}$ is ρ -semilinear on V_1 for an automorphism ρ of $L = K[m_1]$ that coincides with σ^t on K , fixes $m_1 = T^s$ and hence has the same order s/t as σ^t .

STEP 3. Most of the proof now focuses on the semilinear transformation $T_1 := T^t|_{V_1}$ of V_1 , rather than on T and V .

The map T_1 acts irreducibly on the K -space V_1 . For, let W_1 be a nonzero T_1 -invariant subspace of V_1 . Then $W_i := T^{i-1}(W_1)$ is a subspace of V_i for $1 \leq i \leq t$, and $T(W_t) = T^t(W_1) = T_1(W_1) = W_1$. By (3.2), $W_1 \oplus \dots \oplus W_t$ is a nonzero T -invariant subspace of V , and hence $W_1 = V_1$, as required.

STEP 4. By Step 2, T_1 is semilinear on $(V_1)_L$ with associated field automorphism ρ of order $n := s/t$. The ‘polynomial algebra’ $L[T_1]$ (see [4]) is not commutative if $\rho \neq 1$. This leads us to consider the set R of polynomials $f(x) = \sum_0^d x^j f_j$ with $f_j \in L$, using the twisted product $x^j a = a^{\rho^j} x^j$ for $a \in L$. Then R is a (noncommutative) L -algebra having $L[T_1]$ as a homomorphic image under the substitution $x \mapsto T_1$. Jacobson [4] viewed V as an R -module, but we will not need this point of view. We only need to know that each $f \in R$ has a degree in the usual manner, and that $f(T_1)(v_1) = \sum_0^d T_1^j f_j(v_1)$, where $T_1^j f_j$ is a composition of additive maps on V_1 . Then $f(T_1)$ is an additive map on V_1 , but it need not be K -semilinear.

STEP 5. Let $0 \neq f(x) = \sum_0^d x^j f_j \in R$, $f_j \in L$, $f_d = 1$, have minimal degree d such that $f(T_1)(V_1) = 0$. Then $d \leq n$ since $(T_1^n - m_1 I)(V_1) = (T^s - m_1 I)(V_1) = 0$ (by the definition of m_1 in Step 2). We claim that $d = n$; in fact we will show that $f(x) = x^n - m_1$.

Take $a \in L$ lying in no proper subfield, so that $a \neq a^{\rho^j}$ for $0 < j < n$. Consider $g(x) := a^{\rho^d} f(x) - f(x)a \in R$. On the one hand,

$$g(T_1)(v_1) = a^{\rho^d} f(T_1)(v_1) - f(T_1)(av_1) = a^{\rho^d} 0 - 0 = 0.$$

On the other hand, calculating in R we find that

$$g(x) = \sum_0^d (a^{\rho^d} x^j) f_j - \sum_0^d x^j (f_j a) = \sum_0^d x^j (a^{\rho^{d-j}} - a) f_j$$

has degree $< d$ since $a^{\rho^{d-d}} - a = 0$. Now $g(T_1)(V_1) = 0$ and our choice of $f(x)$ imply that $(a^{\rho^{d-j}} - a)f_j = 0$ for $0 \leq j < d$. Then $f_j = 0$ for $0 < j < d$ (since $a^{\rho^{d-j}} \neq a$), so that $f(x) = x^d + f_0$. If $d < n$ then $a^{\rho^{d-0}} \neq a$, so that $f_0 = 0$, whereas $T_1^d(V_1) \neq 0$. Thus, $d = n$ and $f(x) = x^n + f_0$. Finally, since $(f_0 + m_1)(V_1) = 0$ we have $f_0 = -m_1$, as claimed.

STEP 6. We next claim that V_1 has dimension one as a vector space over L . Since T_1 acts irreducibly on V_1 by Step 3, it suffices to exhibit a 1-dimensional subspace of $(V_1)_L$ fixed by T_1 .

By Step 2, $m_1 \in F := C_L(\rho)$, where $[L : F] = |\rho| = s/t = n$. Consequently, if $N_{L/F} : L \rightarrow F$ is the norm map, then there exists an element $a \in L$ such that $N_{L/F}(a) := \prod_0^{n-1} a^{\rho^j}$ equals m_1^{-1} .

Since $h(x) := \sum_0^{n-1} (ax)^j \in R$ has degree less than n , by Step 5 we have $h(T_1)(V_1) \neq 0$. Let $v \in V_1$ with $w := \sum_0^{n-1} (aT_1)^j(v) \neq 0$. Then

$$(aT_1)^n(v) = N_{L/F}(a)T_1^n(v) = N_{L/F}(a)m_1v = v,$$

and hence

$$(aT_1)(w) = \sum_1^{n-1} (aT_1)^j(v) + (aT_1)^n(v) = \sum_1^{n-1} (aT_1)^j(v) + v = w.$$

Thus, $T_1(Lw) = Lw$, so that Lw is the required T_1 -invariant 1-space over L , and hence $\dim(V_1)_L = 1$.

STEP 7. Finally, we need to show that the action of T_1 on V_1 determines T up to $\text{GL}(V)$ -conjugacy. For, if $\mathbf{B} := \{v_{i1} \mid i = 1, \dots, d\}$ is a K -basis of V_1 and $v_{ij} := T^{j-1}(v_{i1})$, then $\{v_{ij} \mid i = 1, \dots, d\}$ is a K -basis of V_j for $1 \leq j \leq t$. If A is the matrix of $T_1 = T^t|_{V_1}$ with respect to \mathbf{B} then

$$v_{1i} \mapsto v_{2i} \mapsto \dots \mapsto v_{ti} \mapsto Av_{1i}, \quad 1 \leq i \leq d,$$

uniquely describes T up to $\text{GL}(V)$ -conjugacy. □

Observe that, in the notation of Steps 1 and 2, $|K| = |C_K(\sigma)|^s = |E|^{nt}$, so $|V| = |L|^t = (|K|^{\deg \mu_1})^t = |E|^{nt^2 \deg \mu_1}$.

PROOF OF THEOREM 1.1. We are given a vector space V of size $n = p^r$ over the prime field $\text{GF}(p)$. We will imitate the preceding proposition in order to construct semilinear transformations over subfields of $\text{End}(V)$ that include all irreducible ones but also include many others. Thus, we will need a decomposition (3.2), a subfield L of $\text{End}(V_1)$ implicit in the statement of Proposition 3.1, a field K , automorphisms of K and L (see Step 2 of the proposition), and a semilinear transformation $T_1 = T^t|_{V_1}$ on V_1 .

Choose a factorization $r = te$ with $e > 1$ and $t|e$; the number of these factorizations is the number $\tau(r) - 1$ of positive divisors of r other than 1. Fix a decomposition (3.2) of V into subspaces V_i of size p^e . Fix a subfield $L \cong \text{GF}(p^e)$ of $\text{End}(V_1)$, so that V_1 is an L -vector space. Given e , all such decompositions and fields are $\text{GL}(r, p)$ -conjugate.

Choose a subfield $K \neq \text{GF}(p)$ of L .

Choose $1 \neq \sigma' \in \text{Aut}(L)$ such that $\sigma'|_K \neq 1$ has order divisible by t . Let $\rho := \sigma'^t$. (Thus, $\sigma := \sigma'|_K \in \text{Aut}(K)$ and $\rho \in \text{Aut}(L)$ satisfy $\sigma^t = \rho|_K$, as required in Step 2 of the proof of Proposition 3.1. According to that step we should also require that $|\sigma^t| = |\rho|$, but we will ignore this restriction in our estimates.)

Extend the action of K from V_1 in order to make V and all V_i vector spaces over K . All such extensions are $\text{GL}(r, p)$ -conjugate.

Choose $\ell \in L^*$, and let $T_1 \in \text{End}(V_1)$ be $v \mapsto \ell v^\rho$, $v \in V_1$ (see Proposition 3.1). We can restrict the choice of ℓ as follows. If $M_a: v \mapsto av$, $a \in L^*$, then $M_a^{-1}T_1M_a: v \mapsto \ell a^{\rho^{-1}}v^\rho$. Since we require different conjugacy classes of transformations T_1 , we can restrict ℓ to a set $\Lambda(e, \rho)$ of

$$|L^*/(L^*)^{\rho^{-1}}| = |C_{L^*}(\rho)| = p^{e/|\rho|} - 1$$

coset representatives of $(L^*)^{\rho^{-1}}$ in L^* .

Up to conjugacy in $\text{GL}(r, p)$, the choices made above uniquely determine $T_1 = T^t|_{V_1}$, and hence also T by the last part of Proposition 3.1. (However, we emphasize that a σ -semilinear map obtained in this manner need not be irreducible on V_K .) Thus, the number of $\text{GL}(r, p)$ -conjugacy classes of pairs K, T , with T an irreducible K -semilinear transformation that is not linear is at most

$$\sum_{e|r, e \neq 1} \sum_{\sigma' \neq 1} |\Lambda(e, \sigma^t)| (\#K \subseteq L, \sigma'|_K \neq 1). \tag{3.3}$$

There are $\tau(r) - 1$ choices for e and L , and then at most $e - 1$ choices for σ' , at most $\tau(e) - 1$ subfields K , and $p^{e/|\rho|} - 1$ elements in $\Lambda(e, \rho)$, where again $\rho = \sigma'^t$. Clearly, $p^{e/|\rho|} - 1$ dominates the corresponding term in (3.3). This is at most $p^{r/3} - 1$ unless σ' has order 2 and either

- (i) $|L| = p^r, t = 1, \rho = \sigma'$ has order 2 and $|\Lambda(e, \rho)| = p^{r/2} - 1$; or
- (ii) $|L| = p^{r/2}, t = 2, \rho = 1, |\sigma'| = 2$ and $|\Lambda(e, \rho)| = p^{r/2} - 1$.

Possibilities (i) and (ii) together contribute at most $2(p^{r/2} - 1)(\tau(r) - \tau(r/2))$ to (3.3). Then (3.3) is easily bounded as required in the theorem if r is not too small, leaving a few cases to be handled by a slightly more detailed and tedious examination of (3.3). □

4. Concluding remarks

We conclude with some elementary observations concerning the semifields \mathcal{S}_T and our arguments.

REMARK 4.1. Note that $\pi_{kT} \cong \pi_T$, $\pi_{T+kI} \cong \pi_T$ and $\pi_{T^{-1}} \cong \pi_T$ for all $k \in K^*$, since $\mathcal{S}_{kT} = \mathcal{S}_T$, $\mathcal{S}_{T+kI} = \mathcal{S}_T$ and $\mathcal{S}_{T^{-1}}T^{d-1} = \mathcal{S}_T$. Thus, as in the desarguesian case, there are isomorphisms among the planes π_T that do not arise from conjugate semilinear transformations.

REMARK 4.2. As in Section 2, if we fix $0 \neq e \in V$ then we obtain a presemifield operation on V from \mathcal{S}_T via $a * b = g(a)(b)$, $a, b \in V$, using the additive isomorphism $g: V \rightarrow \mathcal{S}_T$ defined by $g(A(e)) = A$ for $A \in \mathcal{S}_T$. Then

$$A(e) * v = A(v) \quad \text{for all } A \in \mathcal{S}_T, v \in V,$$

gives a simple description of our operation. In fact, this turns V into a semifield with identity element e , since $e * v = I(e) * v = I(v)$ and $A(e) * e = A(e)$ for all v and A .

REMARK 4.3. It is straightforward to extend the action of L in Proposition 3.1 from V_1 to all of V so as to make all V_i into 1-dimensional L -spaces. However, as has been pointed out to us by Dempwolff via an example [3], there can be irreducible semilinear transformations over K that are not semilinear over any such extension field L .

Nevertheless, a simple way to obtain a candidate for an irreducible σ -semilinear map on a vector space V over a field K is to use σ -semilinearity together with the requirement

$$T: v_1 \mapsto v_2 \mapsto \dots \mapsto v_t \mapsto mv_1 \tag{4.1}$$

for some basis $\{v_1, \dots, v_t\}$ of V and some $m \in K$. If $t > 1$ in (4.1), it is easy to check that the corresponding semilinear map has no invariant 1-space if and only if $m \notin K^{1+\sigma+\dots+\sigma^{t-1}}$. In this case, if $t = 2$ then the corresponding semifield was discovered by Knuth [7].

REMARK 4.4. Similarly, we can obtain many irreducible semilinear transformations by assuming σ -semilinearity in (4.1).

PROPOSITION 4.2. *Let V be a vector space over K with basis v_1, \dots, v_t , and let $\sigma \in \text{Aut}(K)$ and $\rho = \sigma^t$. If $m \in K$ with $m^{\sigma^j-1} \notin K^{\rho-1}$ for $1 \leq j < t$, then (4.1) defines an irreducible σ -semilinear transformation on V with associated field automorphism σ .*

PROOF. Suppose that W is a nonzero T -invariant subspace of V . Let $0 \neq \sum_1^t k_i v_i \in W$, $k_i \in K$, with the minimum number of $k_i \neq 0$. Using T we may assume that $k_1 = 1$. By (4.1) and the fact that W is T^t -invariant,

$$T^t \left(\sum_1^t k_i v_i \right) - m \sum_1^t k_i v_i = \sum_2^t (k_i^\rho m^{\sigma^{i-1}} v_i - k_i m v_i)$$

lies in W and has smaller support, and hence is zero. Then $k_i^\rho m^{\sigma^{i-1}} = k_i m$ for $2 \leq i \leq t$. If some such $k_i \neq 0$ then $m^{\sigma^{i-1}-1} = 1/(k_i^{\rho-1})$, contradicting our condition on m .

Thus, $v_1 \in W$. Applying T shows that all $v_i \in W$, so that $W = V$. \square

REMARK 4.5. We conclude with a very elementary but weaker version of Theorem 1.1 (compare to [6, Theorem 6.2]) having a less informative proof.

PROPOSITION 4.3. *Given a vector space V of size n over a prime field $\text{GF}(p)$, there are fewer than $n \log_p^2 n$ conjugacy classes of pairs (K, T) consisting of a field $K \subseteq \text{End}(V)$ over which V is a vector space and an irreducible semilinear transformation T on V_K .*

PROOF. Let $d = \dim_K V$. Let T be an irreducible σ -semilinear transformation of V_K . Fix a nonzero vector v . Then $\{T^i(v) \mid 0 \leq i < d\}$ is a basis of V (in Section 2 we saw that $\sum_0^{d-1} k_i T^i(v) = 0$, $k_i \in K$, implies that all $k_i = 0$).

Write $T^d(v) = \sum_0^{d-1} k_i T^i(v)$ with $k_i \in K$. Since $T^i(kv) = k^{\sigma^i} T^i(v)$ for each i and each $k \in K$, the k_i completely determine T .

Thus, T is determined by the following choices: a field $K = \text{GF}(p^e)$ over which V is a vector space, an automorphism σ of K , and a choice of $d = r/e$ elements $k_i \in K$, where $|V| = p^r$. There are at most r divisors e of r , at most e choices for σ , and then $|V|$ choices for the k_i . Choosing a K -basis of V amounts to conjugating in $\text{GL}(V_K)$ and hence in $\text{GL}(r, p)$. Consequently, the number of $\text{GF}(p)$ -conjugacy classes of pairs (K, T) is less than $rr|V| = |V| \log_p^2 |V|$, as required. \square

Unlike in the proof of Proposition 3.1, this argument used all $|K|^d = p^r$ possible d -tuples (k_1, \dots, k_d) , which is independent of the choice of K and σ .

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