On the Lenz-Barlotti Classification of Projective Planes

By

Christoph H. Hering and William M. Kantor *)

The purpose of this note is to indicate a uniform proof for the following results.

Theorem 1. *Let $\mathfrak{P}$ be a finite projective plane. Suppose that $\mathfrak{P}$ contains a line r and a point $R \notin r$ such that $\mathfrak{P}$ is (X, XR)-transitive for each point X on r. Then $\mathfrak{P}$ is desarguesian.*

Theorem 2. *Let $\mathfrak{P}$ be a finite projective plane. Suppose that, for some line r and some point $R \in r$, there exists a $1-1$ map σ of the set of points on r other than R onto the set of lines through R other than r such that $\mathfrak{P}$ is (X, X^{σ})-transitive whenever $R \neq X \in r$. Then $\mathfrak{P}$ is desarguesian.*

Theorem 1 implies that there does not exist any finite projective plane of Lenz type III. This completes results of Lüneburg [11 and 12], Cofman [2], Yaqub [17], and Hering [6]. Theorem 2 implies that there does not exist any finite projective plane of Lenz-Barlotti type I.6 or II.3. This result is due to Yaqub [15 and 16], Jónsson [9], Lüneburg [10], and Cofman [3].

Both theorems will be proved by means of some recent results on finite 2-transitive permutation groups due to Shult [14] and Hering, Kantor and Seitz [7]. We remark that, although induction was used in some of the above papers on planes of types III and I.6, there is greater freedom to employ induction in the proofs of the theorems on permutation groups than in the purely geometric situations.

One way of obtaining Theorems 1 and 2 from [14] and [7] is to consider the involutions in the permutation group induced on r by the group G generated by all the (X, XR)- or (X, X^{σ})-perspectivities. This approach requires the investigation of various special situations, and even and odd order planes must be handled differently. We have chosen to use a different approach which provides a more uniform proof.

Proof of Theorem 1. Let G be the group of automorphisms generated by all elations of $\mathfrak{P}$ whose centers lie on r and whose axes contain R. Furthermore, let $Z = G(R, r)$. Then G leaves invariant R and r, and Z is the kernel of the representation of G on the line r. Since G is 2-transitive on r, we have $|G| = (n+1)\, n\, h\, |Z|$, where n is the order of $\mathfrak{P}$ and h is some integer. By Lüneburg [11, p. 441]

$$(n+1)\, n\, (n-1) \,\big|\, |G|\,.$$

*) Research supported in part by NSF Grant GP 9584.

Hence we have

(*). $$\frac{n-1}{(h, n-1)} \,\Big|\, |Z| \,\Big|\, (n-1)\,.$$

Let X be a point on r, x an elation with center X and axis XR, and $z \in Z$. Then the commutator $[x, z] = x^{-1}z^{-1}xz$ leaves invariant all points on the lines r and XR, so that $[x, z] = 1$ and $Z \subseteqq \mathfrak{Z}G$. Let p be a prime dividing $|Z|$ and P a p-Sylow subgroup of Z. Assume that $p \nmid (n+1)\,nh$. Then P is a normal Hall subgroup of G, and by the Schur-Zassenhaus theorem, there exists a complement $\bar{G}$ of P in G. This complement must be normal as $Z \subseteqq \mathfrak{Z}G$. Hence, $\bar{G}$ contains the groups $G(X, XR)$ with $X \in r$, since these groups have order n, which is prime to $|G : \bar{G}|$. This implies that $\bar{G} = G$, a contradiction. Thus, $p \,|\, ((n+1)\,nh, n-1) \,|\, 2h$. Together with (*) we obtain

(**) *If p is a prime and $p \,|\, n-1$, then $p \,|\, 2h$.*

By [14] and [7], we have one of the following possibilities for the 2-transitive group G/Z: (i) a sharply 2-transitive group; (ii) $PSL(2, q)$; (iii) $Sz(q)$; (iv) $PSU(3, q)$; or (v) a group of Ree type. Moreover, in each case G/Z acts on r in its usual 2-transitive permutation representation.

If G/Z is sharply 2-transitive, then $h = 1$ and by (**) $n - 1$ is a power of 2. If $n > 2$, then n is odd and $n + 1$ also must be a power of 2. This implies that $n \leqq 3$, and we have case (ii).

In the unitary case $n = q^3$ and $h \,|\, q^2 - 1$. By (**) every prime divisor of $(n-1)/(q-1) = q^2 + q + 1$ divides $2(q^2 - 1)$. As $(q^2 + q + 1, 2(q^2 - 1)) \,|\, 3$ and $3^2 \nmid q^2 + q + 1$, this is impossible. In the remaining cases $n = q^a$, where $a = 1, 2$, or 3 and q is the order of the ground field (in case (v) the ground field is defined by means of the centralizer of an involution, which is isomorphic to $Z_2 \times PSL(2, q)$ for some power q of 3). Also, $h \,|\, q - 1$ and $((n-1)/(q-1), 2h) = 1$. By (**) this implies that we have case (ii).

Now $n = q$ and $h = (q-1)/d$, where $d = (2, q-1)$. Hence $d \,\big|\, |Z| \,\big|\, n - 1$ by (*). If $q \leqq 3$, it is easy to check that $G \cong SL(2, q)$. Assume that $q > 3$. Then G/Z is simple. Thus $G'Z = G$ and $|G : G'| \,\big|\, |Z| \,\big|\, n - 1$. This implies that G' contains the groups $G(X, XR)$ with $X \in r$ and hence that $G' = G$. Therefore, G is a homomorphic image of the covering group of G/Z. From results of SCHUR [13] and the fact that $(|Z|, q) = 1$, it follows that $G \cong SL(2, q)$. Now we can apply a theorem of LÜNEBURG [11, Satz 2] in order to prove that $\mathfrak{P} \cong PG(2, q)$.

Proof of Theorem 2. Let G be the group generated by the given homologies and let $Z = G(R, r)$. As before, G leaves invariant R and r. Let g be an element of G which fixes all points on r. If g moves some line through R, say X^σ, then $\mathfrak{P}$ is $(X, X^{\sigma g})$-transitive and, therefore, desarguesian by a result of BARLOTTI [1]. We can thus assume that Z is the kernel of the representation of G on $r - \{R\}$. If the order n of $\mathfrak{P}$ is at least 3, then G is 2-transitive on $R - \{r\}$, so that $|G| = n(n-1)\,h\,|Z|$, where h is some integer. We may assume that $n \geqq 5$ (see [4, p. 144, Theorem 15]). Then by LÜNEBURG [10, Satz 2] G is transitive on the set of points which do not lie on r. Therefore, $n^2 \,\big|\, |G|$ and

$$\frac{n}{(h, n)} \,\Big|\, |Z| \,\Big|\, n\,.$$

As in the proof of Theorem 1, we see that $Z \subseteqq \mathfrak{Z} G$. Once again, [14] and [7] imply that the 2-transitive group G/Z is of one of the types (i)—(v) described on page 222.

Assume that G/Z is sharply 2-transitive on $r - \{R\}$. Then $h = 1$, so that $|Z| = n$. Furthermore, G/Z contains an elementary abelian normal p-subgroup P/Z which is sharply transitive on $r - \{R\}$. It follows that $|P/Z| = |Z| = n = p^s$ for some s. Since $P' \subseteqq \Phi(P) \subseteqq Z \subseteqq \mathfrak{Z} P$, we have $[a, bc] = [a, b][a, c]$ for all $a, b, c \in P$. Hence $[a, b]^p = [a, b^p] = 1$ for all $a, b \in P$, and P' is elementary abelian.

Let $a_1 Z, \ldots, a_s Z$ be a set of generators of P/Z, and let $x, y \in P - Z$. Since G acts transitively on $P/Z - \{1\}$, there exists an element $g \in G$ such that $x^g Z = a_1 Z$. Then $[x, y] = [x, y]^g = [x^g, y^g] = [a_1, y^g]$. However, $[a_1, y^g] \in \langle [a_1, a_i] \mid 1 \leqq i \leqq s \rangle$, so that $|P'| \leqq p^{s-1}$. Thus, there exists a maximal subgroup Q of Z containing P'. Since G acts transitively on $P/Z - \{1\}$ and centralizes Z, P/Q is not cyclic. Therefore, $\Omega_1(P/Q) \supsetneqq Z/Q$ and hence $\Omega_1(P/Q) = P/Q$, i.e., P/Q is elementary abelian. The representation of G/P on P/Q is completely reducible. Hence G/Q splits over Z/Q, and G has a normal subgroup N of index p. As $p \nmid n - 1$, N must contain the homology groups $G(X, X^\sigma)$ with $X \in r - \{R\}$. This implies that $N = G$, a contradiction.

Let G/Z be of type (ii)—(v), but $G/Z \not\cong P\Gamma L(2, 8)$. Then G/Z is simple. This implies that $G = G'Z$ and $|G : G'| \mid |Z|$. Since $|G(X, X^\sigma)| = n - 1$ for $X \in r - \{R\}$, we have $G' = G$. Hence Z is a homomorphic image of the Schur multiplier M of G/Z, so that

$$\frac{n}{(h, n)} \Big| |M| .$$

Let q, a and d be defined as in the proof of Theorem 1. Then in the unitary case, $(h, n) \mid (q^2 - 1, q^3 + 1) = q + 1$ and $(q^2 - q + 1) \mid |M|$. In the remaining cases,

$$(h, n) \Big| (q - 1, q^a + 1) = (q - 1, 2) = d \quad \text{and} \quad \frac{1}{d}(q^a + 1) \Big| |M| .$$

If $a = 1$ then $G/Z \cong PSL(2, q)$ and $\frac{1}{d}(q + 1) \nmid |M|$ by Schur [13], a contradiction. Let $a > 1$. A further result of Schur [13] implies that $|M|$ is not divisible by primes p for which G/Z has cyclic p-Sylow subgroups. Therefore, G/Z is not $Sz(q)$, and since $q > 2$ and $q^2 - q + 1 \not\equiv 0 \pmod 9$, it is also not $PSU(3, q)$. For odd primes p dividing $q + 1$ a group of Ree type has cyclic p-Sylow subgroups. Since $q \equiv 3 \pmod 8$ this implies that $q = 3$.

Hence we can assume that $G/Z \cong P\Gamma L(2, 8)$. Let H be the last term of the commutator series of G. Then $H/H \cap Z \cong PSL(2, 8)$, and hence $H \cong PSL(2, 8)$ by Schur [13]. As H is transitive on $r - \{R\}$, we have $H \cdot G(X, X^\sigma) = G$ for $X \in r - \{R\}$. But the order of $H \cdot G(X, X^\sigma)$ is not divisible by n^2, a contradiction.

References

[1] A. Barlotti, Le possibili configurazioni del sistema delle coppie punto-retta (A, a) per cui un piano grafico risulta (A, a)-transitivo. Boll. Un. Mat. Ital. **12**, 212—226 (1957).

[2] J. Cofman, On a characterization of finite desarguesian projective planes. Arch. Math. **17**, 200—205 (1966).

[3] J. Cofman, On the non-existence of finite projective planes of Lenz-Barlotti type I. 6. J. Algebra **4**, 64—70 (1966).

[4] P. DEMBOWSKI, Finite Geometries. Berlin, Heidelberg, New York 1968.
[5] W. GASCHÜTZ, Zur Erweiterungstheorie endlicher Gruppen. J. reine angew. Math. **190**, 63–107 (1952).
[6] C. HERING, Über projektive Ebenen vom Lenz-Typ III. Math. Z. **105**, 219–225 (1968).
[7] C. HERING, W. M. KANTOR and G. M. SEITZ, Finite groups with a split BN-pair of rank 1 (to appear in J. Algebra).
[8] B. HUPPERT, Endliche Gruppen I. Berlin, Heidelberg, New York 1967.
[9] W. J. JÓNSSON, Transitivität und Homogenität projektiver Ebenen. Math. Z. **80**, 269–292 (1963).
[10] H. LÜNEBURG, Endliche projektive Ebenen vom Lenz-Barlotti Typ I-6. Abh. Math. Sem. Univ. Hamburg **27**, 75–79 (1964).
[11] H. LÜNEBURG, Charakterisierungen der endlichen desarguesschen projektiven Ebenen. Math. Z. **85**, 419–450 (1964).
[12] H. LÜNEBURG, Zur Frage der Existenz von endlichen projektiven Ebenen vom Lenz-Barlotti-Typ III.2. J. reine angew. Math. **220**, 63–67 (1965).
[13] J. SCHUR, Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen. J. reine angew. Math. **132**, 85–137 (1907).
[14] E. SHULT, On a class of doubly transitive groups (to appear in Illinois J. Math.).
[15] J. C. D. SPENCER, On the Lenz-Barlotti classification of projective planes. Quart. J. Math. **11**, 241–257 (1960).
[16] J. C. D. S. YAQUB, On projective planes of Lenz-Barlotti class I.6. Math. Z. **95**, 60–70 (1967).
[17] J. C. D. S. YAQUB, The non-existence of finite projective planes of Lenz-Barlotti class III.2. Arch. Math. **18**, 308–312 (1967).

Eingegangen am 22. 7. 1970

Anschrift der Autoren:
Christoph H. Hering
William M. Kantor
Department of Mathematics
University of Illinois at Chicago Circle
Chicago, Ill., USA