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Planes in which every quadrangle lies on a unique Baer subplane

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Abstract Desarguesian projective planes of square order are characterized by the property that every quadrangle lies on a unique Baer subplane.

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1 Introduction

The study of projective planes in terms of their subplane structure goes back at least to work of Ruth Moufang in 1931 (Dembowski, Finite geometries, 1997, 3.4.17), where the equivalence of the little hexagonality condition with the property that every quadrangle generates a Pappian prime subplane is established. Another notable highlight of this stream of thought is Gleason's 1956 theorem [3, 3.4.23] that a finite projective plane in which every quadrangle lies on a Fano subplane is Desarguesian (of even order). More recently, Blokhuis and Sziklai showed in 2000 that a projective plane of order the square of a prime in which every quadrangle lies on a unique Baer subplane is Desarguesian (Blokhuis and Sziklai, Geom. Dedicata, 79:341–347, 2000). (Blokhuis, personal communication October 25, 2010, confirms that the word "unique" must be added to the statement of the theorem as published, in order for the proof to be valid.) The purpose of this article is to reach the same conclusion as Blokhuis

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and Sziklai with a weaker hypothesis on the order of the plane. Like the Blokhuis–Sziklai result, ours will be dependent upon the classification of finite simple groups. More detailed historical background on work of this nature appears in [2, Sect. 2].

2 Background results

The first result we need is the uniqueness of the projective plane of order 4, established by Veblen and Wedderburn in 1907 [8, pp. 387–388]. (N.B.–In [3, 3.2.15] this result is incorrectly attributed to Mac Innes [6], who states but does not prove the result, although he does prove the uniqueness of the plane of order 5.)

Theorem 1 *There is a unique projective plane of order* 4, *up to isomorphism.*

Given two lines L, M of a projective plane π and a point x of π, incident with neither L nor M, the *perspectivity* with centre x from L to M is defined as the map taking the point y of L to the point $xy \cap M$ of M. A *projectivity* is any product of perspectivities.

A *Baer subplane* of a projective plane of order n^2 is a subplane of order n. A *Baer subline* is the intersection of a Baer subplane with a line meeting it in more than one point. A *quadrangle* is a set of four points, no three collinear.

The second result we need is [2, Lemma 3.2], translated from an affine to a projective setting. This is what they actually proved, although it is stated differently there. In particular, the primality hypothesis on the order of the Baer subplane was in no way used. For clarity, we provide a proof, which closely follows the trail they blazed.

Theorem 2 *Let π be a finite projective plane with the property that every quadrangle lies on a unique Baer subplane. Then the set of Baer sublines of π is closed under projectivities.*

Proof We will initially work in an affine plane of order n^2 in which every triangle a, b, c is in a unique affine subplane $A[a, b, c]$ of order n. Any $c' \in [a, b, c]\backslash ab$ determines $A[a, b, c'] = A[a, b, c]$. We will focus on the *affine sublines* of the form $B_r[u, v] := uv \cap A[r, u, v]$. Let L_∞ be the line at infinity.

In our first three steps we will focus on a line M and points $u \notin M$ and $r \in M$. □

Step 1. If $v \neq u$ and uv is parallel to M, then the pair u, v determines a partition $\{M \cap A[s, u, v] \mid s \in M\}$ of M into n parts of size $n^2/n = n$.

Proof Each $A[s, u, v]$ has a line through s parallel to uv and hence meets M in a subline. Any point of that subline determines the same subplane $A[s, u, v]$ together with u, v and hence determines the same subline, producing a partition. □

Step 2. Consider $u \neq w \in B_r[u, v]$, so that $w \in uv$. Then $B_r[u, v] = B_r[u, w]$, and u, w determine the partition $\{M \cap A[s, u, w] \mid s \in M\}$ of M. Since $w \in B_r[u, v]$, *the* u, v *and* u, w *partitions of* M *share the member* $(*) M \cap A[r, u, w] = M \cap A[r, u, v]$.

We now come to the main use of the relationship between the orders of our plane and its subplanes, using the pigeonhole principle.

Step 3. $B_r[u, v] = B_{r'}[u, v]$ if $r' \in M\backslash A[u, v, r]$.

Proof Let $u \neq w \in B_r[u, v]$ and $W \in \{M \cap A[s, u, w] \mid s \in M\}$ with $r \notin W$. Then $|W| = n$ and, by $(*)$, W is disjoint from $M \cap A[r, u, w] = M \cap A[r, u, v]$. Since there are only $n - 1$ other n-point sublines $M \cap A[s, u, v]$ in the u, v-partition of M, one of these sublines, V say, is such that $V \cap W$ contains two points $t \neq t'$. Then $A[u, v, t] = A[u, t, t'] = A[u, w, t]$, so that $V = M \cap A[u, v, t] = M \cap A[u, w, t] = W$. Thus, the $n-1$ members of the u, w partition not containing r belong to the u, v partition. Hence, these two partitions coincide:

$$\{M \cap A[s, u, w] \mid s \in M\} = \{M \cap A[s, u, v] \mid s \in M\}.$$

In particular, $M \cap A[r', u, v]$ and $M \cap A[r', u, w]$ both contain r' and hence coincide. Clearly $M \cap A[r', u, v]$ and u determine $A[r', u, v]$, while $M \cap A[r', u, w]$ and u determine $A[r', u, w]$, so that $A[r', u, v] = A[r', u, w]$.
Thus, $u \neq w \in B_r[u, v] \Rightarrow w \in uw \cap A[r', u, w] = uv \cap A[r', u, v] = B_{r'}[u, v]$. Consequently, $B_r[u, v] = B_{r'}[u, v]$.
We now use projective subplanes containing L_∞. We assume that L_∞ and some of its points are added (when needed) to affine subplanes or sublines being studied. □

Step 4. Consider distinct points x, u, v and lines K, L_∞, M with $x, u, v \in K$ and $x \in K, L_\infty, M$. Then *any two Baer subplanes containing all* 6 *of these objects intersects K in the same set of $n+1$ points.* Dually, *any two such subplanes contain the same set of $n+1$ lines through x.*

Proof Each of the Baer subplanes contains a point $r \neq x$ in M, say, and the desired intersection is $B_r[u, v]$ using $K = uv$ in Step 3. □

Step 5. Given K, L_∞, M, x, u, v as in Step 4 *and $x \neq r \in M$, if $x \neq z \in L_\infty$ then there is a Baer subplane containing r, $B_r[u, v]$ and z* - namely, the projectivization of $A[u, v, s]$ with $s = M \cap uz$, in view of Step 3.

Lemma 1 *Let K, L_∞, M be distinct lines on a point z, let $\mathbf{k} \subset K$ be a subline containing z, and let $z \neq x \in L_\infty$ Then the image of $\mathbf{k}$ under the perspectivity with centre x from K into M is a subline of M.*

Proof By Step 5 there is a Baer subplane B containing $\mathbf{k}, L_\infty, x$. Let L_∞, K', M' be distinct lines of B through x. We will use the last part of Step 4 for x, K', M', L_∞ and two different further pairs of points: one pair in K' and one in M'. (N.B.–The awkward problem addressed by this lemma is that $\mathbf{k} = K \cap B$ where B has nothing to do with M.)

Pick distinct $u, v \in K' \cap B$.

Let $a := M \cap M'$ and $B_1 := A[u, v, a]$, which contains $uv = K'$, x and hence $ax = M'$.

Let $a \neq b \in M' \cap B_1$.

Let $w := K' \cap M$. Then $B_2 := A[w, a, b]$ contains $wa = M$, as well as $ab = M'$, hence x, hence also $xw = K'$.

Apply Step 2 to the sextuple $x, u, v, K', M', L_\infty$ in both B and B_1 to see that {lines of B_1 on x}= {lines of B on x}=$\{xd \mid d \in \mathbf{k}\}$ since $x \in B$ and $\mathbf{k}$ is a line of B.

Apply Step 2 to the sextuple $x, a, b, K', M', L_\infty$ in both B_1 and B_2 to see that {lines of B_2 on x}= {lines of B_1 on x}, which we have just seen is $\{xd \mid d \in \mathbf{k}\}$ and where the lines of B_2 on x meet M in a subline $M \cap B_2$.

Thus, the lines of B_2 on x meet K in $\mathbf{k}$ and meet M in a subline contained in B_2, which proves the lemma.

Lemma 1 establishes that, in a finite projective plane with every quadrangle on a unique Baer subplane, when K, M are distinct lines meeting in a point z, x is a point not on K or M, and $\mathbf{k}$ is a (projective) Baer subline of K containing z, then the image of $\mathbf{k}$ under the perspectivity with centre x from K to M is a Baer subline of M. We now turn to the case where the Baer subline does not contain the point of intersection of the lines K, M. □

Lemma 2 *Let K, M be distinct lines, $z \in K, M$ and $x \notin K, M$. If $\mathbf{k} \subset K$ is a Baer subline not on z, then the image $\mathbf{m}$ of $\mathbf{k}$ under the perspectivity with centre x from K to M is a subline of M.*

Proof Let $a \in \mathbf{k}$, $b \in \mathbf{m}$, such that x is not on the line $N = ab$. Then the perspectivity with centre x from K to M is the product $\tau\sigma$ of the perspectivity σ with centre x from K to N and the perspectivity τ with centre x from N to M. Since $\mathbf{k}$ contains $a = K \cap N$, $sigma(\mathbf{k})$ is a Baer subline of N by Lemma 1 (using $L_\infty = ax$). Also $b \in \sigma(\mathbf{k})$ since $\tau(b) = b \in \mathbf{m} = \tau(\sigma(\mathbf{k}))$. Now $\mathbf{m} = \tau(\sigma(\mathbf{k}))$ is a Baer subline by Lemma 1 (this time using $L_\infty = bx$).

Combining Lemmas 1 and 2, we see that projectivities of our projective plane send Baer sublines to Baer sublines, and we have established Theorem 2.

If L is a line of the projective plane π, then Π_L denotes *the group of all projectivities from L to itself*. We need the main result of Grundhöfer [4] (see also [1, 2.2.4]), augmented by [7] (which ruled out the possibility of the Mathieu group M_{24} of degree 24 occurring for a projective plane of order 23). This result depends on the classification of finite simple groups. □

Theorem 3 *Let L be any line of a finite non-Desarguesian projective plane. Then the group Π_L of projectivities of L contains the alternating group on L.*

Remark 1 If L is a line of $\mathrm{PG}(2, q)$, then Π_L is permutationally isomorphic to $\mathrm{PGL}(2, q)$ acting naturally on $\mathrm{PG}(1, q)$ [1, 2.2.3]. Thus the converse of Theorem 3 holds for finite projective planes of order greater than 4, as $\mathrm{PGL}(2, q)$ does not contain $\mathrm{Alt}(q + 1)$ for $q > 4$.

3 The main result

Theorem 4 *Let π be a finite projective plane of square order. Then π is Desarguesian if and only if every quadrangle lies on a unique Baer subplane.*

Proof Suppose π is non-Desarguesian of order n^2, that L is a line of π, that x and y are distinct points of L and that every quadrangle of π lies on a unique Baer subplane. Then by Theorem 3, the group Π_L of projectivities of L contains the alternating group, and so every subset of L of size $n + 1$ is a Baer subline, by Theorem 2. Thus the number of Baer sublines of π in L and on x and y is the number $C(n^2 - 1, n - 1)$ of subsets of size $n - 1$ of a set of size $n^2 - 1$. But the number of Baer subplanes of π on x and y is

$$n^4(n^2 - 1)^2/n^2(n - 1)^2 = n^2(n + 1)^2,$$

so the number of Baer sublines of π in L and on x and y is at most $n^2(n + 1)^2$ (taking into account sublines lying in more than one Baer subplane). Thus $C(n^2 - 1, n - 1) \le n^2(n + 1)^2$, which forces n to be at most 3.

In fact, for each choice of subline $\mathbf{k}$ containing u, v, and each $x \in \mathbf{k}\backslash\{u, v\}$ along with distinct lines uv, M, L_∞ on x, by Step 3 in the preceding section $\mathbf{k}$ is in n projective subplanes B containing $\mathbf{k}, M, L_\infty$. Pick one B and, for each point $x' \neq x$ of $\mathbf{k}$, distinct lines uv, M', L'_∞ of B on x'. These also lie in n subplanes containing k, but B is the only one containing M (i. e., containing $u, v, M \cap M', M \cap L'_\infty$). Thus, we obtain $n + 1$ sets of n subplanes containing $\mathbf{k}$, any two sets having only B in common: each $\mathbf{k}$ is contained in at least $(n+1)n-1$ subplanes. Now $C(n^2-1, n-1) \le n^2(n+1)^2/[(n+1)n-1]$ implies that $n \neq 3$. If $n = 2$ then Theorem 1 produces a contradiction. The converse is straightforward. □

Remark 2 The argument about the number of subplanes on a subline should be compared with [2, paragraph after Lemma 3.1]. That a plane of order 9 is Desarguesian if every quadrangle is in a proper subplane was already established by Killgrove [5] in 1964. The exception in the proof for $n = 2$ is forced by the fact that $\mathrm{PGL}(2, 4) = \mathrm{Alt}(5)$, and so related to the failure of the converse of Theorem 3 for a plane of order 4.

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