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Some locally finite flag-transitive buildings

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1. INTRODUCTION

This paper is a continuation of Kantor [4, 6] and Ronan [10]. We will construct finite and infinite 'geometries' admitting flag-transitive groups and having buildings as universal covers. The first building is that of $\Omega^+(6, \mathbb{Q}_2)$; the remaining ones are new and strange. As in the aforementioned papers, our motivation is the search for finite geometries that strongly resemble buildings, and for group-theoretic situations similar to BN-pairs.

The paper is split into two independent parts. In section 2 we construct finite GABs (geometries that are almost buildings [4, 6]) with diagram  and rank 2 residues PG(2, 2). One of these is described for each odd integer $m > 1$. Each of them is covered by the $\Omega^+(6, \mathbb{Q}_2)$ building.

The GABs in Section 2 admit flag-transitive groups, and hence can be constructed as coset spaces exactly as in [6, sect. 2]. Consequently, all the relevant terminology is found in that reference (but compare 3.1 below).

In Section 3 we will construct chamber systems (Tits [13], Ronan [9]) whose diagrams are complete graphs and whose rank 2 residues are PG(2, 2) or PG(2, 8). This section is relatively trivial. It is of interest only as an indication of the existence of pathological locally finite flag-transitive buildings.

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2. THE GABs 

2.1. A RATIONAL ORTHOGONAL GROUP

Equip $\mathbb{Q}^7$ with the usual inner product, and let $u_1, \ldots, u_7$ be the standard orthonormal basis. Set $u_* = \Sigma_1^7 u_j$ and $V = u_*^{\perp}$.

The vectors $a_i = -7u_i + u_*$, $1 \leqslant i \leqslant 7$, span V and satisfy the conditions

$$\sum_1^7 a_j = 0, \quad (a_i, a_i) = 42, \quad \text{and} \quad (a_i, a_j) = -7 \quad \text{for} \quad i \neq j. \tag{1}$$

Clearly, there is a subgroup $G_{(a)} \cong A_7$ of GL(V) preserving $\{a_1, \ldots, a_7\}$ (in fact, there is an S_7 as well).

Next, set

$$b_i = a_i + a_{i+1} + a_{i+3} \quad \text{for} \quad 1 \leqslant i \leqslant 7, \tag{2}$$

where subscripts are taken mod 7. Then it is straightforward to check that

$$\sum_1^7 b_j = 0, (b_i, b_i) = 2\cdot 42, \quad \text{and} \quad (b_i, b_j) = 2(-7) \quad \text{for} \quad i \neq j.$$

By (1), there is a linear transformation φ on V such that $a_i^{\varphi} = b_{7-i}$ for all i, and $(u^{\varphi}, v^{\varphi}) = 2(u, v)$ for all $u, v \in V$. Moreover, by (1), (2) and a simple calculation $b_i^{\varphi} = 2a_{7-i}$. Also, $(G_{(a)})^{\varphi} = G_{(b)}$ induces A_7 on $\{b_1, \ldots, b_7\}$.

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There is an obvious PG(2, 2) whose points are the a_i and whose lines are the triples appearing in (2). Thus, $G_{(a)(b)} = G_{(a)} \cap G_{(b)} \cong \mathrm{PSL}(3, 2)$. Moreover, φ induces an outer automorphism of this group.

Let r be the reflection in $(a_1 - a_2)^{\perp}$. Then r induces the transposition (a_1, a_2) on $\{a_1, \ldots, a_7\}$. Set $c_i = b_i^r$. Then $b_1 = c_1$, $b_3 = c_3$ and $b_4 = c_4$. Set $G_{(c)} = (G_{(b)})^r$. Then $G_{(a)(b)(c)}$ is the stabilizer in $G_{(a)}$ of the partition $a_4|a_1a_2|a_3a_6|a_5a_7$. Moreover, the transformation (b_1, b_3, b_4) is in $G_{(b)(c)}$.

$$
\begin{aligned}
b_1 &= 124, & c_1 &= 124, & d_1 &= 2a_6,\\
b_2 &= 235, & c_2 &= 135, & d_2 &= -a_1 + a_2 + a_5 + a_7,\\
b_3 &= 346, & c_3 &= 346, & d_3 &= 2a_4,\\
b_4 &= 457, & c_4 &= 457, & d_4 &= 2a_3,\\
b_5 &= 561, & c_5 &= 562, & d_5 &= a_1 + a_2 + a_5 - a_7,\\
b_6 &= 672, & c_6 &= 671, & d_6 &= a_1 + a_2 - a_5 + a_7,\\
b_7 &= 713, & c_7 &= 723, & d_7 &= a_1 - a_2 + a_5 + a_7.
\end{aligned}
$$

(In the above table we have written $b_1 = 124$ in place of $b_1 = a_1 + a_2 + a_4$. The d_i will be defined soon.) It follows that $G_{(b)(c)} \cong (A_3 \times A_4) \cdot 2$ is the stabilizer of $\{b_1, b_3, b_4\}$ in $G_{(b)}$. Since r normalies $G_{(a)}$, we also have $G_{(a)} \cap G_{(c)} \cong \mathrm{PSL}(3, 2)$.

Now set $d_i = c_i^{\varphi}$ and $G_{(d)} = (G_{(c)})^{\varphi}$. Then $G_{(a)(d)} = (G_{(b)(c)})^{\varphi} \cong (A_3 \times A_4) \cdot 2$ and $G_{(b)(d)} = (G_{(a)(c)})^{\varphi} \cong \mathrm{PSL}(3, 2) \cong G_{(c)(d)}$. Moreover, $G_{(a)(b)(c)(d)} = \langle (a_1, a_2)(a_3, a_6), (a_1, a_5)(a_2, a_7) \rangle \cong D_8$.

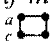
Set $G = \langle G_{(a)}, G_{(b)}, G_{(c)}, G_{(d)} \rangle$. In the notation of [6, sect. 2], $\Gamma = \Delta(G_{(a)}, G_{(b)}, G_{(c)}, G_{(d)})$ *is a GAB in G with diagram*  *where each* •——• *is a* PG(2, 2). In fact, this is clear from the intersections we have just dealt with. (Flag-transitivity follows from [1, (3.10)].)

Note that $G = \langle G_{(a)}, (b_1, b_3, b_4) \rangle$. For, $G_{(b)} = \langle G_{(a)(b)}, (b_1, b_3, b_4) \rangle$, $G_{(c)} = \langle G_{(a)(c)}, G_{(b)(c)} \rangle$ and $G_{(d)} = \langle G_{(a)(d)}, G_{(b)(d)} \rangle$.

Also, $\langle \varphi, r \rangle$ induces a dihedral group of order 8 on $\{G_{(a)}, G_{(b)}, G_{(c)}, G_{(d)}\}$.

2.2. Finite GABs

Clearly, $G_{(a)}\langle r \rangle \cong S_7$ permutes the vectors $a_1, \ldots, a_7$, and hence can be regarded as a group of 6×6 integral matrices with respect to the basis $a_1, \ldots, a_6$ of V (since $a_7 = -\Sigma_1^6 a_i$). Also, φ produces an integral matrix, while $\varphi^{-1} = \frac{1}{2}\varphi$ has all its entries in $\mathbb{Z}[\frac{1}{2}]$. Thus, $G\langle \varphi, r \rangle = \langle G_{(a)}, \varphi, r \rangle \leqslant \mathrm{GL}(6, \mathbb{Z}[\frac{1}{2}])$. If m is any odd integer then we can view all of these 6×6 matrices mod m. Whenever $D \subseteq \mathrm{GL}(6, \mathbb{Z}[\frac{1}{2}])$ let $D^{(m)}$ be the corresponding set of matrices mod m.

The homomorphism $G \to G^{(m)}$ induces an isomorphism on $G_{(a)}$ for each $a \in \{a, b, c, d\}$, and preserves intersections among these four groups. Thus, *if $m > 1$ and m is odd then $\Gamma^{(m)} = \Delta(G_{(a)}^{(m)}, G_{(b)}^{(m)}, G_{(c)}^{(m)}, G_{(d)}^{(m)})$ is a finite GAB with diagram*  *and group $G^{(m)}$*. (Once again, flag-transitivity follows from [2, (3.10)].) Clearly, $G \to G^{(m)}$ induces a cover $\Gamma \to \Gamma^{(m)}$. In 2.3 we will see that this is a universal cover.

Theorem 1. *$G^{(p)} = \Omega^{\pm}(6, p)$ for each prime $p \neq 2, 7$, where $G^{(p)} = \Omega^{+}(6, p)$ if and only if $p \equiv 1, 2$ or 4 (mod 7).*

Proof. Let $u_1, \ldots, u_7, u_8$ be the standard orthonormal basis of $\mathbb{Q}^8$. Then

$$7\left(\sum_1^8 \mathbb{Z}[\tfrac{1}{2}]u_i\right) + \mathbb{Z}[\tfrac{1}{2}]u_* = \left(\sum_1^6 \mathbb{Z}[\tfrac{1}{2}]a_i\right) \oplus \mathbb{Z}[\tfrac{1}{2}]u_* \oplus 7\mathbb{Z}[\tfrac{1}{2}]u_8 \tag{3}$$

since $7u_i = u_* - a_i$ for $1 \leqslant i \leqslant 7$. Moreover, the three summands on the right hand side of (3) are pairwise orthogonal.

Passing mod p we find that $GF(p)^8$ becomes an orthogonal geometry decomposed into the orthogonal sum of a 6-space and a 2-space, each of which is preserved by $G^{(p)}$. The 2-space inherits the quadratic form $7x^2 + 49y^2$, which has nontrivial zeros if and only if $(-7/p) = 1$; that is, if and only if $p \equiv 1$, 2 or 4 (mod 7).

Since $(G_{(a)})' = G_{(a)}$, $G' = G$ and $(G^{(p)})' = G^{(p)}$. Thus, $G^{(p)}$ is contained in the orthogonal group specified in the theorem.

On the other hand, $(G\langle r\rangle)^{(p)}$ lies in $O^{\pm}(6, p)$, and $r^{(p)}$ is a reflection. Thus, $(G\langle r\rangle)^{(p)} \geqslant \Omega^{\pm}(6, p)$ by Wagner [14]. This proves that $G^{(p)} = \Omega^{\pm}(6, p)$, as required.

When $p = 3$, both $\Delta^{(p)}$ and the theorem were first obtained by Aschbacher and Smith [2].

2.3. The $\Omega^+(6, \mathbb{Q}_2)$ Building

Next, we will identify the GAB in (2A):

Theorem 2. *The GAB Γ is isomorphic to the affine building for $\Omega^+(6, \mathbb{Q}_2)$.*

We will imitate the proof given in [6] of a similar result.

Lemma 1. *$V \otimes_{\mathbb{Q}} \mathbb{Q}_2$ is isometric to an $\Omega^+(6, \mathbb{Q}_2)$-space.*

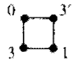
First Proof. By [6, (6.1)], $\mathbb{Q}_2^8$ is an $\Omega^+(8, \mathbb{Q}_2)$-space. By (3), the orthogonal complement of $V \otimes_{\mathbb{Q}} \mathbb{Q}_2$ inherits the quadratic form $7x^2 + 49y^2$, and this has nontrivial zeros in $\mathbb{Q}_2$.

Second Proof. Let $\lambda, \bar{\lambda} = (-1 \pm \sqrt{-7})/2 \in \mathbb{Q}_2$. A straightforward calculation (using the fact that $\lambda^2 + \lambda + 2 = 0$) proves that the vectors $(\lambda, \lambda, \bar{\lambda}, \lambda, \bar{\lambda}, \bar{\lambda}, 3)$, $(3, \lambda, \lambda, \bar{\lambda}, \lambda, \bar{\lambda}, \bar{\lambda})$ and $(\bar{\lambda}, 3, \lambda, \lambda, \bar{\lambda}, \lambda, \bar{\lambda})$ span a totally singular 3-space lying in $V \otimes_{\mathbb{Q}} \mathbb{Q}_2$.

The affine building Δ for $\Omega^+(6, \mathbb{Q}_2)$ can be described as follows (Bruhat–Tits [3]). The vector space $V' = V \otimes_{\mathbb{Q}} \mathbb{Q}_2$ has a basis $e_1, e_2, e_3, f_1, f_2, f_3$ such that all inner products are 0 except for $(e_j, f_j) = (f_j, e_j) = 1$. Define the four $\mathbb{Z}_2$-lattices L_i by

$$\begin{aligned} L_0 &= \langle e_1, e_2, e_3, f_1, f_2, f_3\rangle, \\ L_1 &= \langle \tfrac{1}{2} e_1, e_2, e_3, 2f_1, f_2, f_3\rangle, \\ L_3 &= \langle \tfrac{1}{2} e_1, \tfrac{1}{2} e_2, \tfrac{1}{2} e_3, f_1, f_2, f_3\rangle, \\ L_{3'} &= \langle \tfrac{1}{2} e_1, \tfrac{1}{2} e_2, \tfrac{1}{2} f_3, f_1, f_2, e_3\rangle, \end{aligned}$$

where the brackets denote generation of lattices over the ring $\mathbb{Z}_2$ of 2-adic integers. Let P_i be the stabilizer of L_i in $\Omega^+(6, \mathbb{Q}_2)$. Then $\Delta = \Delta(P_0, P_1, P_3, P_{3'})$. The corresponding diagram is



Note that there is an obvious dihedral group of order 8 inducing graph automorphisms on Δ.

Let $L_a = \langle a_1, \ldots, a_7\rangle$, and define L_b, L_c and L_d similarly.

Lemma 2. *We may assume that L_a is a scalar multiple of L_0.*

Proof. $G_{(a)}$ fixes some point of the realization of Δ (Bruhat–Tits [3, pp. 64–65]). Since $G_{(a)} \cong A_7$ it fixes a vertex. Thus, we may assume that $G_{(a)} < P_0$. Choose $\kappa \in \mathbb{Q}_2$ so that $\kappa a_1 \in L_0 - 2L_0$. Then $L_0 \supseteq \kappa L_a + 2L_0 \supset 2L_0$. By the irreducibility of $G_{(a)}$, $L_0 = \kappa L_a + 2L_0$.

Since $L_0/L_0 \cap \kappa L_a \subseteq 2(L_0/L_0 \cap \kappa L_a)$, $L_0 \subseteq \kappa L_a$ by Nakayama's Lemma [7, p. 242]. Thus, $L_0 = \kappa L_a$.

LEMMA 3. *We may assume that L_b, L_c and $L_a \cap (\frac{1}{2}L_d)$ are scalar multiples of L_3, $L_{3'}$ and $L_0 \cap L_1$, respectively.*

PROOF. By definition, $L_a \supset L_b$. Applying φ, we find that $L_b \supset 2L_a$. Now apply r in order to deduce that $L_a \supset L_c \supset 2L_a$. Also, $L_a \cap (\frac{1}{2}L_d) \supseteq 2L_a$. (For, $L_a \cap (\frac{1}{2}L_d)$ contains a_6, a_4, a_3, $a_1 + a_2$, $a_1 - a_2$, $a_1 + a_5$ and $a_1 - a_5$.)

Thus, the group $S = G_{(a)(b)(c)(d)}$ fixes the following three subspaces of $L_a/2L_a$:

$$\begin{aligned} L_b/2L_a &= \langle b_1 + 2L_a, b_2 + 2L_a, b_3 + 2L_a\rangle \\ L_c/2L_a &= \langle c_1 + 2L_a, c_2 + 2L_a, c_3 + 2L_a\rangle \\ L_a \cap (\tfrac{1}{2}L_d)/2L_a &= \langle a_6 + 2L_a, a_4 + 2L_a, a_3 + 2L_a, a_1 + a_2 + 2L_a, a_1 + a_5 + 2L_a\rangle. \end{aligned}$$

Also, $L_a/2L_a$ inherits the structure of an $\Omega^+(6, 2)$-space (the quadratic form being $v \mapsto (v, v)/2 \pmod 2$). The subspaces $L_b/2L_a$, $L_c/2L_a$ and $(L_a \cap (\frac{1}{2}L_d)/2L_a)^\perp = \langle d_2 + 2L_a\rangle$ are totally singular and pairwise incident (since $b_1 = c_1$ and $b_3 = c_3 = -d_2 - 2a_1$), and hence form a flag of the $\Omega^+(6, 2)$-space $L_a/2L_a$. This is the only such flag fixed by S.

If $L_0 = \kappa L_a$ then S fixes a flag of $\kappa L_a/2\kappa L_a = L_0/2L_0$. Choose notation first so that this flag is $2L_3/2L_0$, $2L_{3'}/2L_0$, $(L_0 \cap L_1/2L_0)^\perp$ and then so that $2L_3 = \kappa L_b$, $2L_{3'} = \kappa L_c$, and $L_0 \cap L_1 = \kappa(L_a \cap (\frac{1}{2}L_d))$.

PROOF OF THEOREM 2. By Lemmas 2 and 3, $G_{(a)} < P_0$, $G_{(b)} < P_3$, $G_{(c)} < P_{3'}$ and $G_{(a)(d)} < P_0 \cap P_1$. Set $r' = r^\varphi$ Then r' interchanges L_a and $\frac{1}{2}L_d$, and hence normalizes the stabilizers $P_0 \cap P_1$ of $L_a \cap (\frac{1}{2}L_d)$. Thus, $G_{(d)} = (G_{(a)})^{r'} < P_0^{r'}$ where $P_0 \cap P_1 < P_0^{r'}$. Since $G_{(d)} \nleq P_0$, it follows that $P_0^{r'} = P_1$. This proves that $G_{(a)} < P_0$, $G_{(b)} < P_3$, $G_{(c)} < P_{3'}$ and $G_{(d)} < P_1$.

It follows that G induces a flag-transitive group on the residue of each of the vertices P_i of Δ. Since Δ is connected, it follows that G is flag-transitive on Δ.

Now define $\Gamma \to \Delta$ via $G_{(a)}g \mapsto P_0 g$ and so on (where $g \in G$). This is a cover. Since Δ is simply connected (Tits [13]), it follows that $\Gamma \cong \Delta$.

REMARK. By considering their discriminants it is easy to show that $L_a = L_0$, $L_b = L_3$, $L_c = L_{3'}$ and $\frac{1}{2}L_d = L_1$.

2.4. THE GROUP G

Let f be the quadratic form $42\Sigma_1^6 x_i^2 - 14\Sigma_{1 \leqslant i < j \leqslant 6} x_i x_j$ obtained from (1) by using the basis $a_1, \ldots, a_6$ of V. Let $\Omega(\mathbb{Z}[\frac{1}{2}], f)$ be the commutator subgroup of the corresponding orthogonal group over $\mathbb{Z}[\frac{1}{2}]$.

THEOREM 3. $G = \Omega(\mathbb{Z}[\frac{1}{2}], f)$.

PROOF. By 2.2, $G \leqslant \mathrm{GL}(6, \mathbb{Z}[\frac{1}{2}])$. Also, $G = G'$ and G preserves the form f obtained by restricting to V the usual form on $\mathbb{Q}^7$. Thus, $G \leqslant \Omega = \Omega(\mathbb{Z}[\frac{1}{2}], f)$.

By Theorem 1 and Lemma 1, Ω acts flag-transitively on Δ. Then $\Omega = (\Omega \cap P_0)G$. We will show that $\Omega \cap P_0 \leqslant G$.

Let $g \in \Omega \cap P_0$. The matrix (x_{ij}) of g with respect to the basis $a_1, \ldots, a_6$ must have all entries x_{ij} in $\mathbb{Z}[\frac{1}{2}]$. On the other hand, Lemma 2 implies that $(L_a)^g = L_a$, so that $x_{ij} \in \mathbb{Z}_2$. Thus, $x_{ij} \in \mathbb{Z}$. Since g preserves f, there are only finitely many possibilities for (x_{ij}).

The centralizer C of $L_a/2L_a$ in $\Omega \cap P_0$ is a finite subgroup normalized by $G_{(a)}$, and $C \cap G_{(a)} = 1$. Then C consists of scalars. But $-1 \notin \Omega$ (since $-1 \notin \Omega^-(6, 17) = \Omega^{(17)}$). Thus, $C = 1$ and $\Omega \cap P_0$ is isomorphic to a subgroup of $\Omega^+(6, 2)$ containing $G_{(a)} \cong A_7$. Since $\Omega^+(6, 2) \cong A_8$ is not isomorphic to any subgroup of $\mathrm{GL}(6, \mathbb{Q})$, it follows that $\Omega \cap P_0 = G_{(a)} < G$, as required.

2.5. Further Properties

We conclude this section with several remarks concerning G and Γ.

2.5.1. Assume that $G^{(p)} = \Omega^{\pm}(6, p)$ with $p \equiv \pm 1 \pmod 4$. Then $-1 \in G^{(p)}$, while $-1 \notin G^{(p)}_{(x)}$ for each x. Let 'bar' denote the homomorphism $G^{(p)} \to G^{(p)}/\langle -1 \rangle$. Then $\bar{\Gamma}^{(p)} = \Delta(\bar{G}^{(p)}_{(a)}, \bar{G}^{(p)}_{(b)}, \bar{G}^{(p)}_{(c)}, \bar{G}^{(p)}_{(d)})$ is a finite GAB with group $\bar{G}^{(p)}$, and there is an obvious cover $\Gamma^{(p)} \to \bar{\Gamma}^{(p)}$. *This is a 2-fold cover.* For, -1 acts nontrivially on $\Gamma^{(p)}$ since -1 does not fix $G^{(p)}_{(a)}$.

Note that this is quite different from the situation in [6], where -1 always acted trivially.

2.5.2. By construction, $\langle \varphi, r\rangle^{(m)}$ induces a D_8 of graph automorphisms of $\Gamma^{(m)}$.

If $p \neq 2$ is a prime and $(2/p) = 1$ then $\langle \varphi, r\rangle^{(p)} < G^{(p)}$: the graph automorphisms are induced by inner automorphisms. In particular, for such primes p all $G^{(p)}_{(x)}$ are conjugate in $G^{(p)}$.

2.5.3. In the case $p = 7$ not covered by Theorem 1, $G^{(7)} = 7^5\Omega(5, 7)$. For, if $a_i' = a_i/7$ then G acts on

$$\sum_1^6 \mathbb{Z}[\tfrac{1}{2}]a_i' = \sum_2^6 \mathbb{Z}[\tfrac{1}{2}](a_1' - a_i') + \mathbb{Z}[\tfrac{1}{2}]a_1',$$

while $a_1' - a_7' = -\Sigma_2^6(a_1' - a_i') + 7a_1'$ and $(a_1' - a_i', a_1' - a_j')$ is 1 if $i \neq j$ and 2 if $i = j$. Passing mod 7 we obtain a 6-space V_6 over GF(7) and a hyperplane V_5 upon which $G^{(7)}$ acts. Moreover, V_5 inherits a nontrivial G-invariant inner product (although V_6 does not). Using $(G\langle r\rangle)^{(7)}$ as in the proof of Theorem 1, we find that G induces $\Omega(5, 7)$ on V_5. On the other hand, $(G_{(a)})^{(7)}$ does not fix any 1-space of V_6. From this it follows that $G^{(7)}$ cannot act faithfully on V_5, and hence that $G^{(7)}$ is as claimed.

Moreover, the homomorphism $G^{(7)} \to \Omega(5, 7)$ induces a *7^5-fold cover from $\Gamma^{(7)}$ onto a GAB with diagram*  *and group $\Omega(5, 7)$.*

2.5.4. The situation in 2.3 closely resembles that of [6]. This relationship can be made more precise, as follows. Let $u_1, \ldots, u_8$ be as in the proof of Theorem 1. Set $v_* = u_* + u_8 = \Sigma_1^8\, u_i$. Regard $G_{(a)}\langle r\rangle$ as a group of isometries of $\mathbb{Q}^8$ permuting $\{u_1, \ldots, u_8\}$ and fixing u_8. Extend φ to $\mathbb{Q}^8$ by letting $u_8^{\varphi} = \frac{1}{2}v_*$ and $v_*^{\varphi} = 4u_8$. Then $(u^{\varphi}, v^{\varphi}) = 2(u, v)$ for all $u, v \in \mathbb{Q}^8$, so that $G\langle r, \varphi\rangle$ projectively preserves the form $\Sigma_1^8 x_i^2$. Call this form f_8, and set $G_8 = \Omega(\mathbb{Z}[\frac{1}{2}], f_8)$. Then $G = (G_8)_{u_8, v_*}$. Moreover, if r_8 and r_* are the reflections in u_8 and v_*, then both lie in G_8 while $G = C_{G_8}(\langle r_8, r_*\rangle)'$. Finally, the complex Γ in 2.2 can be identified with the set of fixed points of $\langle r_8, r_*\rangle$ on the complex Δ_8 occurring in [6, sect. 5].

2.5.5. Three vectors were introduced in the second proof of Lemma 1; call them v_1, v_2, v_3. Set $v_4 = u_* + \sqrt{-7}\, u_8$. Let $\bar{v}_i$ be defined by replacing $\sqrt{-7}$ by $-\sqrt{-7}$. Then v_1, v_2, v_3, v_4 and $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$ span complementary totally singular 4-spaces of $\mathbb{Q}_2^8$.

2.5.6. As in [6, (10.3)], it can be shown that any finite flag-transitive GAB with diagram arises as an image of Γ in such a way that the commutator subgroup of the given group is a homomorphic image of G. More generally, a similar results holds for chamber-transitive SCABs (cf. 3.1) with the above diagram.

3. Complete Graph Diagrams

The present section is based on a generalization of GABs.

3.1. Chamber Systems and Cosets

A *chamber system* of rank n consists of a set $\mathscr{C}$ of objects called *chambers*, together with a family of partitions $\mathscr{E}_i$ of $\mathscr{C}$, where i ranges through a set I of size n. If $J \subseteq I$ then $\mathscr{E}_J$ denotes the join of the partitions $\mathscr{E}_j$, $j \in J$. The chamber system is *connected* if $\mathscr{E}_I = \mathscr{C}$. A member of $\mathscr{E}_i$ is called an *i-edge*. A *vertex* is a member of $\mathscr{E}_{I-\{i\}}$ for some $i \in I$.

If $x \in \mathscr{E}_J$, $J \subseteq I$, then the *residue* Res(x) is the chamber system consisting of the set of chambers in x, together with the intersections of this set with all $\mathscr{E}_j$, $j \in J$. Clearly, Res(x) has rank $|J|$. A chamber system is *residually connected* if Res(x) is connected whenever $x \in \mathscr{E}_J$ for some J of size $\leqslant n - 2$.

Definition. A chamber system is a *SCAB* (chamber system that is almost a building) if it is residually connected of rank $n \geqslant 3$ and if, for each 2-set $\{i, j\} \subseteq I$, there is an integer n_{ij} such that Res(x) is a generalized n_{ij}-gon for each $x \in \mathscr{E}_{\{i,j\}}$. The *diagram* of the SCAB is defined as follows: it has node set I; the distinct nodes i and j are joined by $n_{ij} - 2$ edges if $n_{ij} \in \{2, 3, 4\}$, by $\frac{1}{2}n_{ij}$ edges if $n_{ij} = 6$ or 8, and by an edge labeled n_{ij} in all other cases.

In this terminology, a GAB can be regarded as a SCAB in which every flag has a nonempty intersection. (Here, a *flag* is a set of vertices of the SCAB any two of which have a nonempty intersection. Compare Tits [11, p. 3] and [13].)

We will only be interested in connected chamber systems that admit an automorphism group G transitive on $\mathscr{C}$. In this situation, fix $C \in \mathscr{C}$, let B be its stabilizer in G, and let E_i be the stabilizer of the i-edge containing C. Then the chamber system is isomorphic to the chamber system $(G, B, E_i)_{i \in I}$ defined as follows: chambers are cosets of B; and each i-edge is a coset of E_i, regarded as a set of cosets of B. Conversely if G is a group generated by a family E_i, $i \in I$, of subgroups, and $B \leqslant \cap\{E_i | i \in I\}$, then the above definition produces a connected chamber system with a chamber-transitive automorphism group. If $E_J = \langle E_j | j \in J \rangle$ then $\mathscr{E}_J$ can be identified with the set of cosets of E_J.

A theorem of Tits [13] states that, if $\mathscr{C}$, $\mathscr{E}_i (i \in I)$ is a SCAB, and if every rank 3 residue having a spherical diagram is a building, then there is a universal 2-covering SCAB $\tilde{\mathscr{C}}$, $\tilde{\mathscr{E}}_i$ $(i \in I)$ that is a building. Thus, there is a surjection $\tilde{\mathscr{C}} \to \mathscr{C}$ such that, whenever $J \subseteq I$ and $|J| \leqslant 2$, every member of $\tilde{\mathscr{E}}_J$ is mapped bijectively onto a member of $\mathscr{E}_J$. Consequently, rank 2 residues are mapped isomorphically. If the original SCAB has the form $(G, B, E_i)_I$ then its universal 2-cover also has a chamber-transitive automorphism group, and hence has the form $(\tilde{G}, \tilde{B}, \tilde{E}_i)_I$, where $G = \tilde{G}/N$ for the group N of covering transformations (compare Ronan [9]). Moreover, $B = \tilde{B}N/N$ and $E_i = \tilde{E}_i N/N$.

In the remainder of this section we will have $B = 1$, so that G will be regular on chambers.

3.2. Frobenius Groups

If a SCAB has as diagram the complete graph K_n on $n = |I|$ vertices, and if each edge corresponds to PG(2, q), we will say that the diagram is K_n/PG(2, q).

Let $q = 2$ or 8.

REMARK. If G is a group generated by a finite set $\{E_i | i \in I\}$ of $n \geq 3$ subgroups of order $q + 1$, any two of which generate a Frobenius group of order $(q^2 + q + 1)(q + 1)$, then $(G, 1, E_i)_I$ is a SCAB with diagram K_n/PG(2, q).

PROOF. If $j \neq k$ then $(E_{\{j,k\}}, 1, E_i)_{\{j,k\}}$ is PG(2, q), since the Frobenius group $E_{\{j,k\}}$ is sharply flag-transitive on that plane. (NB: This flag-transitivity holds only for $q = 2$ or 8.)

3.3. EXAMPLES WITH DIAGRAM K_n/PG(2, q)

As in 3.2, let $q = 2$ or 8. It is easy to give many examples of SCABs using 3.2. The following examples are not GABs. Nevertheless, each has as universal cover a flag-transitive building [13].

EXAMPLE 1. Let $F = \mathrm{GF}(q^2 + q + 1)$ and $m \geq 1$, and let G be the semidirect product of F^m with a cyclic group $\langle t \rangle$ of order $q + 1$, where t induces a scalar transformation on F^m. Then *G produces many SCABs with diagrams* K_n/PG(2, q), $n \leq (q^2 + q + 1)^m$, as follows. Let $v_1 = 0, \ldots, v_n$ be any set of elements generating F^m. Then $\{\langle v_i t \rangle | 1 \leq i \leq n\}$ satisfies the requirements of 3.2. For, if $i \neq j$ then $\langle v_i t, v_j t \rangle = \langle v_i v_j^{-1}, v_i t \rangle$ where $(v_i v_j^{-1})^{v_i t} = v_i^s v_j^{-s} = (v_i v_j^{-1})^s$ for some integer s satisfying $0 < s < q^2 + q + 1$.

Of course, each of these SCABs is a residue of a certain one having diagram K_N/PG(2, q), where $N = (q^2 + q + 1)^m$. In fact, if we allowed infinite index sets I then all of these for all m would be residues of a SCAB having as diagram a countable complete graph.

It is easy to check that these SCABs are never GABs.

Note that if $m = 2 = n - 1$ then the above SCAB has diagram K_3/PG(2, q).

EXAMPLE 2. *The group G of all transformations $x \to ax^\sigma + b$ over $F = \mathrm{GF}(q^3)$, $q = 2$ or 8, $a^{q^2+q+1} = 1$, $b \in F$, $\sigma \in \mathrm{Aut}\, F$, produces a SCAB with diagram* K_3/PG(2, q), as follows. Note that F contains an element α satisfying $\alpha^{q+1} = \alpha + 1$. Then $\alpha^{q^2+q+1} = \alpha(\alpha^{q+1})^q = \alpha(\alpha^q + 1) = 1$. The three groups G_{01}, $G_{0\alpha}$ and $G_{1\alpha}$ have order $q + 1$. For, this is clear in the case of G_{01}, while $G_{0\alpha}$ and $G_{1\alpha}$ are obtained by conjugating by $x \to \alpha x$ and $x \to (\alpha + 1)x + \alpha$, respectively. Then 3.2 applies to $\{G_{01}, G_{0\alpha}, G_{1\alpha}\}$.

This example can be generalized, as follows.

EXAMPLE 3. Let $q = 2$ or 8, let $m \geq 1$, and let $G^{(m)}$ be the group obtained from the direct product of m copies of the group G in Example 2 by identifying all the m versions of $x \to x^2$; thus, $|G| = q^{3m}(q^2 + q + 1)^m(q + 1)$. Then *$G^{(m)}$ produces a SCAB with diagram* K_{2m+1}/PG(2, q). For, let $t \in G^{(m)}$ have order $q + 1$. In each copy of G' pick elements a_i, b_i of order 7 such that Example 2 applies to $\{t, a_i t, b_i t\}$ for $1 \leq i \leq m$. Then 3.2 applies to $\{\langle t \rangle, \langle a_i t \rangle, \langle b_i t \rangle | 1 \leq i \leq m\}$. For, $\langle a_i t, b_j t \rangle = \langle a_i b_j^{-1}, a_i t \rangle$. If $i = j$ our choice of a_i and b_j yields that this is a Frobenius group. If $i \neq j$ then $|a_i b_j^{-1}| = q^2 + q + 1$ and $(a_i b_j^{-1})^{a_i t} = a_i^2 b_j^{-2} = (a_i b_j^{-1})^2$, as required.

Note that the resulting SCAB is a covering of one of the SCABs in Example 1.

EXAMPLE 4. $G = \mathrm{P\Gamma L}(2, q^3)$, $q = 2$ or 8, *produces a SCAB with* $(q^3 + 1)(q - 1)$ vertices and *diagram* K_4/PG(2, q). For, if α is as in Example 2 then 3.2 can be applied to $\{G_{\infty 01}, G_{\infty 0\alpha}, G_{\infty 1\alpha}, G_{01\alpha}\}$ (where the subscripts refer to stabilizers when G is regarded as acting on the projective line $\{\infty\} \cup \mathrm{GF}(q^3)$). That the first three of these behave as desired was shown in Example 2. Any element of PSL(2, q^3) interchanging ∞ and 0 and sending 1 to α must be an involution, and hence leaves invariant our set of four groups. Hence, any three of the groups behave as desired.

Of course, all rank 3 residues are as in Example 2. As in Examples 2 and 3, we can generalize this example as follows.

EXAMPLE 5. Let G be the semidirect product of the direct product of $n \geqslant 1$ copies of $\mathrm{PSL}(2, q^3)$, $q = 2$ or 8, by a cyclic group $\langle t \rangle$ of order $q + 1$, where t induces the field automorphism $x \to x^2$ on each factor. Then *G produces a SCAB with diagram $K_{3n+1}/\mathrm{PG}(2, q)$.* For, let a_i, b_i, c_i be elements of the ith factor such that $\{\langle t \rangle, \langle a_i t \rangle, \langle b_i t \rangle, \langle b_i t \rangle, \langle c_i t \rangle\}$ is as in Example 4. Then the union of these n sets behaves as in 3.2. Namely, using Example 2 we see that $a_i^t = a_i^2$, $b_i^t = b_i^2$ and $c_i^t = c_i^2$. Thus, we can proceed exactly as in Example 3.

Finally, Examples 1, 3 and 5 can be merged in a similar manner:

EXAMPLE 6. Consider the group $(A \times B \times C)\langle t \rangle$ where A, B, C and t are as follows: $A = \mathrm{GF}(q^2 + q + 1)^k$ and $a^t = 2a$ for all $a \in A$; $B\langle t \rangle$ is the group in Example 3; and $C\langle t \rangle$ is the group in Example 5. Then *$(A \times B \times C)\langle t \rangle$ produces many SCABs with diagram $K_N/\mathrm{GF}(q)$,* the largest N being $N = (q^2 + q + 1)^k + 2m + 3n$.

PROBLEM. When can two buildings constructed as in the above examples (via [13]) be isomorphic?

Note that Example 6 shows that any two such buildings arise as residues of one of these buildings.

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