THOSE NASTY BAER INVOLUTIONS

by

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1. Introduction

Baer involutions are big nuisances. They are frequently the major obstacles to the study of collineation groups of finite projective planes. Several years ago it was standard to rig hypotheses so they would not occur. Recently, attempts have been made to deal with them. We will describe a few such attempts.

Probably the most important and successful attempt was the Ostrom-Wagner theorem [18]. Other related results of this sort are discussed in [4], 4.4. More recent trends have been to apply deep classification theorems concerning finite simple groups. The results most used are the Feit-Thompson theorem, the Alperin-Brauer-Gorenstein-Walter theorem [1], the fundamental results of Bender [2] and Shult [21] on permutation groups, along with the extension of Shult's result by Hering [6]. We note that all but one of the major classification theorems concerning finite simple groups involve involutions. That exception is Thompson's theorem on quadratic pairs [22], and that theorem essentially contains as a special case the recent powerful results of Ostrom and Hering [7] on elation groups in translation planes of characteristic $\neq 2,3$.

Several talks in this conference deal with planes using very deep group-theoretic results. Some comments on the necessity of this sort of approach is, perhaps, worthwhile. It is not unreasonable to feel that the use of group theory is a crutch which could be eliminated by a sufficiently detailed geometric study -- and there are, indeed, instances where this may eventually be possible.

For such results as the Ostrom-Wagner theorem, only elementary group theory is ever needed. However, the minute one wishes to study the structure of more general collineation groups (such as an arbitrary simple collineation group) group theory seems unavoidable at present. For example, in his very important characterization of $PSU(3,q)$, O'Nan [16] implicitly has a projective plane of order q^2 available (this is described, somewhat more explicitly, in [13]). Moreover, he has a group looking very much like $PSU(3,q)$, acting on the plane very much like $PSU(3,q)$ acts on $PG(2,q^2)$. Nevertheless, there does not seem to be a geometric way to show the plane is desarguesian, although a geometric proof would be very nice to have.

If it is difficult to show a group is $PSU(3,q)$ given a great deal of information concerning its structure and action on a plane, how can we expect to obtain the structure of a collineation group given comparatively little information (such as, for example, the structure of a Sylow 2-subgroup)?

2. The well-behaved case.

Throughout our discussion, $\mathcal{P}$ will be a projective plane of order n and Γ a collineation group of even order.

In this section, we will indicate how nice things are when all involutions in Γ are perspectivities. For the sake of simplicity, we will assume that Γ is generated by its involutions, and that Γ has no nontrivial solvable normal subgroup of even order. Finally, assume $\Gamma \neq \Gamma(x,X)$ for any x,X. Let $O(\Gamma)$ be the largest normal subgroup of Γ of odd order, and set $\bar{\Gamma} = \Gamma/O(\Gamma)$.

If n is even, the structure of Γ is known by [19] and [2]. In particular, $\bar{\Gamma}$ is $PSL(3,2^e)$, A_6, $PSL(2,2^e)$, $Sz(2^e)$, or $PSU(3,2^e)$ for some e. Moreover, $O(\Gamma)$ is the center $Z(\Gamma)$ of Γ.

We will thus assume that n is odd. The following is the most basic lemma known in the study of involutory homologies.

<u>Lemma 1</u>. ([17], [15], [11].) Let σ and τ be commuting involutory homologies of a finite projective plane having different axes.

(i) $\sigma\tau$ is an involutory homology.

(ii) σ is the only involutory homology having the same center and axis as σ.

(iii) A collineation group containing $\langle\sigma,\tau\rangle$ has no elementary abelian subgroup of order 8 generated by 3 homologies.

Suppose Γ has no σ and τ as in Lemma 1. Then again by [2], $\bar{\Gamma}$ is $PSL(2,2^e)$, $Sz(2^e)$, or $PSU(3,2^e)$. So suppose σ and τ exist. Then Γ has no elementary abelian subgroup of order 8. Now group theory takes over. By [1], $\bar{\Gamma}$ is $PSL(2,q)$, $PGL(2,q)$, $PSL(3,q)$, $PSU(3,q)$, A_7, $PSU(3,4)$, or M_{11}. Actually, M_{11} can be shown not to occur, $PSU(3,4)$ is an unlikely possibility, while $\Gamma = A_7$ occurs for $\mathcal{P} = PG(2,5^2)$. More information is obtained from the following technical lemma.

Lemma 2. Let $\mathcal{P}$ be a projective plane of odd order, and Δ a collineation group of $\mathcal{P}$. Suppose $\Delta/O(\Delta)$ is isomorphic to the alternating group A_4 of degree 4 , and that Δ contains commuting involutory homologies σ and τ having different axes. Then $\langle\sigma,\tau\rangle$ centralizes $O(\Delta)$.

In our situation, $\bar{\Gamma}$ contains many A_4's. It follows readily that $O(\Gamma) = Z(\Gamma)$. From this we get the following possibilities for Γ : PSL(2,q) , a non-split central extension of PSL(2,9) by a group of order 3 , PGL(2,q) , PSL(3,q) , SL(3,q) , PSU(3,q) , SU(3,q) , A_7 , or PSU(3,4) .

Still more partial information is possible when $\bar{\Gamma}$ is PSL(2,q) , PGL(2,q) , PSL(3,q) , or PSU(3,q) , as follows.

If $\bar{\Gamma}$ is PSL(3,q) , then $\mathcal{P}$ has a Γ-invariant desarguesian subplane of order q on which Γ induces $\bar{\Gamma}$. Moreover, $q|n$, $q - 1|n - 1$, and $q + 1|n^2 - 1$. (All of this is also true when q and n are even, and $\bar{\Gamma}$ is $PSL(3,2^e)$.)

If $\bar{\Gamma}$ is PGL(2,q) and $(q,n) \neq 1$, then $\mathcal{P}$ has a desarguesian Γ-invariant subplane of order q on which Γ induces $\bar{\Gamma}$. Moreover, $q|n$, $q - 1|n - 1$, and $q + 1|n^2 - 1$. (All of this is also true when q and n are even, and $\bar{\Gamma}$ is $PGL(2,2^e)$.)

If $\bar{\Gamma}$ is PSU(3,q) and $(q,n) \neq 1$, then $\mathcal{P}$ has a desarguesian Γ-invariant subplane of order q^2 on which Γ induces $\bar{\Gamma}$. Moreover, $q|n$, $q - 1|n - 1$, and $q + 1|n^2 - 1$.

If $\bar{\Gamma}$ is PSL(2,q) or PGL(2,q), and if $3\nmid q$, then Γ has point - and line - orbits of size $q + 1$. This is, however, false when $3|q$; an example is provided by a subgroup PSL(2,9) of PSL(3,25). Similarly, if $\bar{\Gamma}$ is PSU(3,q) and $3\nmid q$, then Γ has point - and line - orbits of size $q^3 + 1$. In each of these cases, Γ induces the usual 2-transitive representation of $\bar{\Gamma}$ on its orbits of size $q + 1$ or $q^3 + 1$.

These results on the action of specific groups Γ on a projective plane should be compared with [14], [3], and [10]. There are clearly some open problems even in these concrete situations.

3. Baer involutions

Almost nothing general is known about collineation groups of even order containing no involutory perspectivities. It would be particularly nice to know, for example, that a plane of odd order does not admit a Klein group $\langle\alpha,\beta\rangle$ of collineations such that α,β, and $\alpha\beta$ are Baer involutions. This, or any reasonable facsimile of this, would combine with Lemma 1 to make collineation groups of even order in planes of odd order infinitely more accessible to geometric and group-theoretic analysis. Such Klein groups can, however, occur for planes of even order.

Some useful combinatorial information has recently been found when a Baer involution preserves a regular polarity ([5], [9], [20]). Also, if the Klein group $\langle\alpha,\beta\rangle$ mentioned above happens to preserve a polarity $\mathfrak{G}$, A. Hoffer has pointed out the possibility that the 4 polarities in $\langle\alpha,\beta,\mathfrak{G}\rangle - \langle\alpha,\beta\rangle$ might be played off against one another. But little is known even in this more special situation.

Consider a plane $\mathcal{P}$ of order n and a collineation group Γ of even order. Again suppose that Γ has no nontrivial solvable normal subgroup of even order, but now suppose Γ has Baer involutions. Also, suppose Γ has no nontrivial normal subgroup of the form $\Gamma(x,X)$.

If n is even and Γ contains an involutory elation, the structure of Γ is known by [6]. The same is true if n is odd and Γ contains a Klein group of homologies having the same axis (again by [6]), an oddball situation that probably cannot occur. These two cases are accessible for nongeometric reasons: in both cases, the permutation representation of Γ on the set of centers of involutory perspectivities is of a very special type.

All other cases are open. In particular, even when n is odd and Γ contains an involutory homology, little is known in general. Unlike when n is even, the representation of Γ on the centers of involutory homologies has no special properties, as is seen from the examples $P\Gamma L(2,q)$ and $P\Gamma U(3,q)$. The only tools available as of now are Lemma 1 and the following results [12]; the first has the same flavor as Gleason's lemma.

Lemma 3. Let $\mathcal{P}$ be a plane of odd order, Δ a collineation group of $\mathcal{P}$ fixing a line A and a point $a \in A$, and b a point of $A - a$. If $\Delta(b)$ contains an involutory homology but no Klein group, then Δ is transitive on those points $c \in A - a$ which are centers of (not necessarily involutory) nontrivial homologies.

Lemma 4. Let L and X be lines of a plane of odd order and $x,y \in L$, $x \neq y$. Let Δ be a collineation group satisfying the following conditions.

(i) $\Delta(x,X)$ contains an involutory homology, but no Klein group.

(ii) $\Delta(y)$ contains a nontrivial elation with axis $\neq L$.

(iii) $|\Delta(y)|$ is odd.

(iv) $X \cap L \notin y^{\Delta}$.

Then $\Delta(L)$ contains an involutory homology, and $x^{\delta} = X \cap L$ for some $\delta \in \Delta$.

4. Perspectivities

The following situation is of interest in view of Piper's talk. It also provides a testing ground for working with Baer involutions.

(*) Let $\mathcal{P}$ be a (not necessarily finite) projective plane, and Γ a collineation group fixing no point or line. Suppose that each point of $\mathcal{P}$ is the center of a nontrivial perspectivity in Γ.

The only known examples in which (*) holds are desarguesian or Moufang. However, even then Γ need not contain the little projective group of $\mathcal{P}$. This is a major hazard which must be dealt with when studying (*) . For example, Γ might be $PSU(3,q)$ and $\mathcal{P} = PG(2,q^2)$.

If (*) holds, so does one of the following: (i) if L is a line for which $\Gamma(L) \neq 1$, then $\Gamma(x,L) \neq 1$ for all $x \in L$; or (ii) there is a 1 - 1 mapping θ from the set of points into the set of lines such that $y \in x^{\theta}$ implies $x \in y^{\theta}$, $\gamma\theta = \theta\gamma$ for all $\gamma \in \Gamma$, and $\Gamma(x,X) \neq 1$ if and only if $X = x^{\theta}$. In the case of a finite $\mathcal{P}$, (i) implies that $\mathcal{P}$ is desarguesian and Γ contains the little projective group, while in (ii) θ must be a polarity preserved by Γ . Thus, (i) makes direct contact with Piper's talk: some results stated there are special cases of the situation in (i). Case (ii) suggests the following slightly weaker situation.

(#) θ is a polarity of a finite projective plane $\mathcal{P}$, and Γ is a collineation group preserving θ such that $\Gamma(L) \neq 1$ for each non-absolute line containing at least one absolute point.

This situation actually permits a possibility excluded by (*) : θ can be an orthogonal polarity of a desarguesian plane of odd order.

Assume (#) . Let $\mathcal{P}$ have order n , and let A be the set of absolute points of θ . A major obstacle to the study of (#) is the fact that Γ is not known to be transitive on A -- although even then, and even if (*) holds, we cannot yet show that $\mathcal{P}$ must be desarguesian. The following theorem collects the known conditions under which (#) implies that $\mathcal{P}$ is desarguesian.

<u>Theorem</u>. Assume (#) . Then $\mathfrak{P}$ is desarguesian if any one of the following holds.

(1) Γ contains an involutory perspectivity.

(2) n is not a fourth power.

(3) θ induces an orthogonal polarity on the fixed plane of each Baer involution in Γ .

(4) Γ is transitive on those nonabsolute lines which contain absolute points.

Actually, (2), (3) and (4) are fairly easy consequences of (1). The proof of (1) was, in effect, accomplished in two steps. In [11], the case where θ is a unitary polarity was handled using quite a lot of group theory; Baer involutions had to be considered in great detail. The general case of (1) was then completed in [12], using arguments which are more geometric. Lemmas 1 and 3 were basic for the proof.

The only other information known about (#) is as follows. $\mathfrak{P}$ is desarguesian if θ is orthogonal. We may thus assume that θ is not orthogonal, and hence that n is a square. If $n > 4$, then Γ has no nontrivial solvable normal subgroup. (Surprisingly, this is not easy to prove.) By the Feit-Thompson theorem, this at least provides involutions to play with.

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