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ON POINT-TRANSITIVE AFFINE PLANES

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ABSTRACT

Finite affine planes are constructed admitting nonabelian sharply point-transitive collineation groups. These planes are of two sorts: dual translation planes, and planes of type II.1 derived from them.

1. Introduction

In [5], Ostrom used the dual Tits–Lüneburg planes in order to construct affine planes of type II.1. In this note, we will construct translation planes, point-transitive (affine) dual translation planes, and point-transitive affine planes of type II.1. The derivation process involved in the construction of the last of these planes is a standard, straightforward imitation of Ostrom's approach. On the other hand, the translation planes we use behave differently from those used by Ostrom. In §2, a construction is given for translation planes of order q^2 having kernel GF(q) and admitting an abelian group of order q^2 which has an orbit of length q^2 at infinity but contains only q elations. This abelian group is elementary abelian if and only if q is odd. Our construction was motivated by examples in [3, (4.5)]; these and other examples are presented in §§3, 5.

The corresponding dual translation planes and derived dual translation planes of type II.1 appear in §4. *One plane of each sort is obtained whenever* $q > 2$ *and* $q \equiv 2 \pmod 3$, *and one more whenever* $q = 5^e > 5$. The full collineation group of each of these planes is determined. This group is transitive on the q^4 points but has no line-orbit of length q^4. In particular, *the corresponding projective planes are not self-dual,* and none is isomorphic to the dual of any other. Consequently, still further planes of type II.1 arise by duality. (The same proofs apply to the derived dual Tits–Lüneburg planes, the determination of whose collineation groups was

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left open in Ostrom [5]. Since these groups again act differently on the planes and their duals, duality produces still further planes of type II.1.)

The planes studied in §4 are point-transitive affine planes which are not translation planes. Finite planes with these properties seem to be rare (cf. Dembowski [1, pp. 183–184, 214–215]). Moreover, *each of these planes admits a sharply point-transitive group.*

I am grateful to Jill Yaqub for directing my attention to Johnson and Piper [2]. Those authors obtained planes of type II.1 by deriving the duals of translation planes of order q^2 constructed by Walker [6] whenever $q \equiv 5 \pmod 6$. It is easy to check that the planes constructed in those papers are precisely the planes $\mathscr{A}(l)$ and $\mathscr{A}(l)'$ considered here for which q is odd and $l = 0$.

All of our proofs are straightforward except, perhaps, at the end of §4. Most of the prerequisites can be found on pp. 132, 226 and 249–251 of Dembowski [1].

2. The planes $\mathscr{A}(l)$

Set $K = \mathrm{GF}(q)$, where $q > 3$ and $q = p^e$ is a power of a prime $p \neq 3$.

DEFINITION. A function $l: K \to K$ is *likeable* if it satisfies the conditions:
(i) $l(t+u) = l(t) + l(u)$ for all $t, u \in K$, and
(ii) if $u^2 = t^2u - \frac{1}{3}t^4 + tl(t)$ then $t = u = 0$.

Throughout this section, l will denote a likeable function. Property (i) and a calculation yield the following result.

LEMMA 2.1. *Let $f(t,u) = tu - \frac{1}{3}t^3 + l(t)$. Then the q^2 matrices*

$$M(t,u) = \begin{pmatrix} 1 & t & u & f(t,u) \\ 0 & 1 & t & u \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $t, u \in K$ form an abelian group $P(l)$. If q is odd then $P(l)$ is elementary abelian. If $p = 2$ then $P(l)$ is the direct product of e cyclic groups of order 4.

DEFINITION. Let $\Sigma(l)$ consist of the following 2-dimensional subspaces of K^4:

$$0 \times 0 \times K \times K$$

$$(K \times K \times 0 \times 0)M, \qquad M \in P(l).$$

PROPOSITION 2.2. *$\Sigma(l)$ is a spread.*

PROOF. It suffices to check that $(K \times K \times 0 \times 0) \cap (K \times K \times 0 \times 0)M(t, u) = 0$ when t or u is nonzero. But this requires that the equations

$$xu + yt = 0$$
$$xf(t, u) + yu = 0$$

have only the trivial solution $x = y = 0$, and hence that $u^2 - tf(t, u) \neq 0$. This is guaranteed by the definition of likeability.

PROPOSITION 2.3. (i) *$P(l)$ has an orbit of length q^2 on the line L_∞ at infinity.*
(ii) *The elations in $P(l)$ are the matrices of the form $M(0, u)$.*

PROOF. The first assertion is obvious, and the second is easily checked. (In fact, if $t \neq 0$ then $M(t, u)$ fixes only q vectors.)

THEOREM 2.4. *$\Sigma(l)$ determines a nondesarguesian translation plane $\mathcal{A}(l)$.*

PROOF. This is clear by (2.3).

COROLLARY 2.5. *The group $N(l) = \mathrm{GL}(4, q)_{\Sigma(l)}$ fixes the point ∞ common to L_∞ and $0 \times 0 \times K \times K$.*

LEMMA 2.6. *$P(l)$ is a Sylow p-subgroup of $N(l)$.*

PROOF. Some Sylow p-subgroup of $N(l)$ has the form $P(l)B$, where B fixes both $0 \times 0 \times K \times K$ and $K \times K \times 0 \times 0$. Then B consists of matrices of the form

$$\begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and a simple calculation completes the proof.

THEOREM 2.7. *The planes $\mathcal{A}(l)$ and $\mathcal{A}(l')$ are isomorphic if and only if $l'(t) = l(t^{\sigma^{-1}}\gamma^{-\sigma^{-1}})^\sigma\gamma^3$ or $p = 2$ and $l'(t) = l(t^{\sigma^{-1}}\gamma^{-\sigma^{-1}})^\sigma\gamma^3 + t\beta^2 + t^2\beta$ for some $\sigma \in \mathrm{Aut}\, K$, some $\gamma \in K^*$, some $\beta \in K$, and all $t \in K$.*

PROOF. Let $S \in \Gamma\mathrm{L}(4, q)$ send $\Sigma(l)$ to $\Sigma(l')$. We may assume that S fixes $0 \times 0 \times K \times K$ and $K \times K \times 0 \times 0$ (by (2.3i)) and conjugates $P(l)$ to $P(l')$ (by (2.6)). Then S has the form $vS = v^\sigma S'$ for some $\sigma \in \mathrm{Aut}\, K$ and some matrix S' of the form

$$S' = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \alpha' & \beta' \\ 0 & 0 & 0 & \gamma' \end{pmatrix}$$

with $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in K$.

Define l^σ by $l^\sigma(t) = l(t^{\sigma^{-1}})^\sigma$. Since $P(l)^\sigma = P(l^\sigma)$, we can replace l by l^σ in order to have $S = S'$.

Since S sends elations of $\mathcal{A}(l)$ to elations of $\mathcal{A}(l')$, (2.3ii) and a simple calculation yield that

$$c\begin{pmatrix}\alpha & \beta \\ 0 & \gamma\end{pmatrix} = \begin{pmatrix}\alpha' & \beta' \\ 0 & \gamma'\end{pmatrix} \quad \text{for some } c \in K^*.$$

By replacing T by $\alpha^{-1}T$ we may assume that $\alpha = 1$. Computing $S^{-1}M(t,u)S$, we find that $t\gamma = ct\gamma^{-1}$, $c(u - t\beta\gamma^{-1}) = c(u + t\beta\gamma^{-1})$ and $f'(t\gamma, cu + ct\beta\gamma^{-1}) = c\{f(t,u) - t\beta^2\gamma^{-1}\}$ for all t, u. The theorem now follows easily.

COROLLARY 2.8. *$N_{N(l)}(P(l))/P(l)K^*$ is isomorphic to the group of all matrices*

$$\begin{pmatrix} 1 & \beta & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma^2 & \gamma^2\beta \\ 0 & 0 & 0 & \gamma^3 \end{pmatrix}$$

such that $l(t) = l(t\gamma^{-1})\gamma^3$ and $\beta = 0$, or $l(t) = l(t\gamma^{-1})\gamma^3 + t\beta^2 + t^2\beta$ and $p = 2$, for all $t \in K$.

LEMMA 2.9. (i) *$P(l)$ fixes each line $0 \times 0 \times K \times K + (0,0,0,d)$ of the desarguesian Baer subplane $\mathcal{A}_0 = 0 \times K \times 0 \times K$.*

(ii) *$\mathcal{A}_0$ has q images under $P(l)$.*

(iii) *The group $P(l)K^*$ generated by $P(l)$ and the dilatations with center 0 has 3 orbits of lines parallel to $0 \times 0 \times K \times K$, of lengths 1, $q-1$ and $q^2 - q$.*

PROOF. The required calculations are straightforward.

3. Example of likeable functions

In this section we will present two examples of likeable functions.

LEMMA 3.1. *An additive function $l: K \to K$ is likeable if and only if the equation*

$$x^2 - x + \tfrac{1}{3} - l(a)/a^3 = 0$$

has no solution for $x \in K$ and $a \in K^*$. In particular, if q is odd then l is likeable if and only if $l(a)a^{-3} - 1/12$ is a nonsquare for all $a \in K^*$.

PROOF. Set $a = t$ and $x = u/a^2$ in the definition of likeability.

EXAMPLE I. The constant function $l = 0$ is likeable if and only if $q \equiv 2 \pmod 3$. In §5 we will see that $\mathscr{A}(l)$ arose in [3]. Note that (by (2.8) or a simple calculation) $\operatorname{diag}(1, r, r^2, r^3) \in \operatorname{Aut} \mathscr{A}(l)$ for each $r \in K^*$.

EXAMPLE II. Let $q = 5^e > 5$ and fix a nonsquare $k \in K$. Then $l(t) = kt^5 + k^{-1}t$ is likeable (since $l(t)t^{-3} - 1/12 = k^{-1}t^{-2}(kt^2+1)^2$ and $t^2 \neq -k^{-1}$). Different nonsquares produce isomorphic planes. By (2.7), these planes are different from those of Example I.

REMARKS. (1) If q is even and $q \equiv 2 \pmod 3$, let $T : K \to \mathrm{GF}(2)$ be the trace map. Then $\operatorname{Ker} T = \{y \in K \mid y = x^2 + x \text{ for some } x \in K\}$ and $T(1) = 1$. Thus, $l(t)$ is likeable if and only if $T(l(t)/t^3) = 0$ for all $t \neq 0$. Consequently, the set of likeable functions is closed under addition.

(2) If $q \equiv 1 \pmod 3$ then $x^2 + x + \frac{1}{3} = 0$ has a root, and hence $\operatorname{Ker} l = 0$.

4. Dual and derived dual planes

Let $\mathscr{A}(l)$ be as in §2. Let V be the translation group of $\mathscr{A}(l)$ (so $V \cong K^4$), and let $V(\infty)$ consist of those translations whose center ∞ is the parallel class of $0 \times 0 \times K \times K$. Note that $|VP(l)| = q^6$ and $|V(\infty)P(l)| = q^4$. Since $P(l)$ is transitive on $L_\infty - \{\infty\}$, while $V(\infty)$ is transitive on the affine lines through each point of $L_\infty - \{\infty\}$, $V(\infty)P(l)$ is transitive on the lines not containing ∞.

Let $\mathscr{A}(l)^*$ denote the projective plane dual to the projective closure of $\mathscr{A}(l)$. We will use $L_\infty^* = \infty$ as its line at infinity in order to regard $\mathscr{A}(l)^*$ as an affine plane.

PROPOSITION 4.1. *$P(l)V(\infty)$ is a nonabelian group sharply transitive on the affine points of $\mathscr{A}(l)^*$; it contains exactly q^3 translations. Moreover, $\mathscr{A}(l)^*$ is not a translation plane.*

PROOF. This is straightforward. (Note that the center of $P(l)V(\infty)$ is $\{M(0, u) \mid u \in K\}$.)

Let $\mathscr{A}_0$ be as in (2.9), and let $\mathscr{L}$ consist of L_∞ and the lines in (2.9i). Then $\mathscr{A}(l)^*$ is derivable (Ostrom [4, theorem 9]), and $\mathscr{L}^*$ is a derivation set. The derived plane $\mathscr{A}(l)'$ has the same points as the affine plane $\mathscr{A}(l)^*$; its lines are those of $\mathscr{A}(l)^*$ not meeting $\mathscr{L}^*$, together with all Baer subplanes of $\mathscr{A}(l)^*$ containing $\mathscr{L}^*$.

THEOREM 4.2. *The plane $\mathscr{A}(l)'$ has type* II.1. *It admits a nonabelian sharply point-transitive group containing exactly q^3 translations.*

PROOF. Clearly, $P(l)V(\infty)$ acts sharply transitively, and its q^3 translations appearing in (4.1) constitute all translations of $\mathscr{A}(l)'$. The q^2 translations fixing $\mathscr{A}_0^*$ produce one (c, L)-transitivity (where $L = L'_\infty$ is the new line at infinity, while c is the parallel class ∞' of the new line $\mathscr{A}_0^*$). As in Ostrom [5], we must assume that $\mathscr{A}(l)'$ is a dual translation plane and derive a contradiction.

There are q^2 elations with center ∞' of the (alleged) dual translation plane $\mathscr{A}(l)'$ fixing a Baer subplane of $\mathscr{A}(l)'$ which used to be a point of $\mathscr{A}(l)$ on L_∞. Only q of these elations are translations, but all are inherited by the derived plane $\mathscr{A}(l)^*$ of $\mathscr{A}(l)'$. Thus, there is a group of q collineations of $\mathscr{A}(l)^*$ fixing a Baer subplane pointwise. By (2.6), no such group exists.

THEOREM 4.3. (i) $\mathrm{Aut}\,\mathscr{A}(l)'$ *is inherited from* $\mathrm{Aut}\,\mathscr{A}(l)^*$.

(ii) *If* $\mathscr{A}(l_1)' \cong \mathscr{A}(l_2)'$ *then* $\mathscr{A}(l_1) \cong \mathscr{A}(l_2)$.

(iii) $\mathscr{A}(l)'$ *is not isomorphic to a derived dual Tits–Lüneburg plane or a derived Hughes plane.*

PROOF. Clearly, (i) implies (iii). Also, if (i) holds then $\mathrm{Aut}\,\mathscr{A}(l)'$ has a unique orbit on L'_∞ of length q (by (2.9)), which together with ∞' is a derivation set producing $\mathscr{A}(l)^*$. Thus, (i) also implies (ii), and we only need to verify (i).

By (2.6) a Sylow p-subgroup of $\mathrm{Aut}\,\mathscr{A}(l)^*$ has order q^6e' with $e' \mid e$; and this has a subgroup Q_1 of order q^5e' acting on $\mathscr{A}(l)'$. Clearly, $Q_1 \geqq P(l)V(\infty)B$, where B is a group of q elations of $\mathscr{A}(l)^*$ which fix both 0 and $\mathscr{A}_0^*$. The axis L of B becomes a Baer subplane (also called L) of $\mathscr{A}(l)'$, and B is a group of collineations of $\mathscr{A}(l)'$ fixing this subplane pointwise.

Any collineation of $\mathscr{A}(l)'$ fixing L must fix its set of points at infinity. The latter points form the derivation set D for $\mathscr{A}(l)'$ such that the corresponding derived plane is $\mathscr{A}(l)^*$. Thus, $(\mathrm{Aut}\,\mathscr{A}(l)')_L \leqq \mathrm{Aut}\,\mathscr{A}(l)^*$. In particular, the centralizer of B lies in $\mathrm{Aut}\,\mathscr{A}(l)^*$.

If $\mathrm{Aut}\,\mathscr{A}(l)'$ fixes D then (4.3) holds, so assume that D is moved. By (2.9), we already know orbits of lengths $1, q, q^2 - q$ on L'_∞. Thus, $\mathrm{Aut}\,\mathscr{A}(l)'$ is transitive on $L'_\infty - \{\infty\}$. Its Sylow p-subgroups then have order $\geqq q^6e'$. If $B < Q \in \mathrm{Syl}_p(\mathrm{Aut}\,\mathscr{A}(l)')_0$ then $|Q| \geqq q^2e'$.

Let $1 \neq z \in Z(Q)$. If $z \in B$ then $Q \leqq \mathrm{Aut}\,\mathscr{A}(t)^*$. Thus, $z \notin B$, and $\langle z, B\rangle \leqq \mathrm{Aut}\,\mathscr{A}(l)^*$. Then z fixes the line L of $\mathscr{A}(l)^*$ and centralizes the q elations in B. Since $\langle z, B\rangle$ fixes 0 and $\mathscr{A}_0^*$ it cannot be faithful on $\mathscr{A}_0^*$. By (2.6), $\langle z, B\rangle = B$, which is ridiculous.

REMARK. The same argument settles a question left open in Ostrom [5]: *the automorphism group of a derived Tits–Lüneburg plane is precisely the inherited group.*

5. The planes with $l = 0$

In this section we will show that the planes $\mathscr{A}(l)$ with l identically 0 are the same as those appearing in [3, (4.5)].

Let $q \equiv 2 \pmod 3$. Set $F = GF(q^2)$, $K = GF(q)$, $\bar{\alpha} = \alpha^q$ and $T(\alpha) = \alpha + \bar{\alpha}$ for $\alpha \in F$. Form the K-space

$$V = \{(\alpha, \beta + K, \gamma, b, c) \mid \alpha, \beta, \gamma \in F, T(\gamma) = 0; b, c \in K\}.$$

Equip V with the quadratic form

$$Q(\alpha, \beta + K, \gamma, b, c) = \alpha^2 + \alpha\bar{\alpha} + \bar{\alpha}^2 + T(\beta\gamma) + bc.$$

Then V is an $\Omega^+(6, q)$ space. A spread in K^4 corresponds (under the Klein correspondence) to a set Ω of $q^2 + 1$ singular points of V, no two of which are perpendicular. The set Ω in [3, (4.5)] consists of the points

$$\langle 0, 0, 0, 0, 1 \rangle$$
$$\langle \rho, \rho\bar{\alpha} + K, \sigma, 1, \rho\bar{\rho} \rangle$$

where $T(\sigma) = 0 = T(\rho) + \sigma\bar{\sigma}$.

Proposition 5.1. *The translation plane determined by Ω is $\mathscr{A}(l)$ where $l = 0$.*

Proof. Let $\omega \in F$ and $\omega^3 = 1 \neq \omega$. Set $\theta = 1 + 2\omega$, so $T(\theta) = 0$. Write

$$(\alpha, \beta + K, \gamma, b, c) = (x\omega + \tfrac{1}{3}u\bar{\omega}, -y\omega + K, \tfrac{1}{3}a\theta, \tfrac{1}{3}e, c)$$

with $x, u, y, a, d, e \in K$. Note that

$$Q(x\omega + \tfrac{1}{3}u\bar{\omega}, -y\omega + K, \tfrac{1}{3}a\theta, \tfrac{1}{3}e, c) = \tfrac{1}{3}xu + \tfrac{1}{3}ya + \tfrac{1}{3}ce.$$

Let $\sigma, \rho \in F$ with $T(\sigma) = 0 = T(\rho) + \sigma\bar{\sigma}$. Write $\sigma = \frac{1}{3}a\theta$ and $\rho = x\omega + \frac{1}{3}u\bar{\omega}$, so $0 = -x - \frac{1}{3}u + \frac{1}{3}a^2$ and $\rho\bar{\sigma} \equiv \frac{1}{3}a(x + \frac{1}{3}u) \pmod K$. Then Ω consists of the points

$$\langle 0, 0, 0, 0, 1 \rangle$$
$$\langle (\tfrac{1}{3}a^2 - \tfrac{1}{3}u)\omega + \tfrac{1}{3}u\bar{\omega}, \tfrac{1}{9}a^3\omega + K, \tfrac{1}{3}a\theta, 1, (\tfrac{1}{3}a^2 - u)^2 - (\tfrac{1}{3}a^2 - u)u + u^2 \rangle$$

for $a, u \in K$.

Now identify $(x\omega + \frac{1}{3}u\bar{\omega}, -y\omega + K, \frac{1}{3}a\theta, \frac{1}{3}e, c)$ with the vector (e, a, u, x, y, c) in K^6, and replace Q by $3Q$. Then $Q(e, a, u, x, y, c) = ec + ay + ux$, while Ω consists of the points

$$\langle 0, 0, 0, 0, 0, 1 \rangle$$
$$\langle 1, a, u, a^2 - u, -\tfrac{1}{3}a^3, u^2 - a^2u + \tfrac{1}{3}a^4 \rangle.$$

Under the Klein correspondence, $\langle 1, a, u, x, y, -xu - ya \rangle$ corresponds to the 2-space $\langle (1, 0, -x, y), (0, 1, a, u) \rangle$. This completes the proof of (5.1).

Appendix: Coordinates

Coordinates for $\mathscr{A}(l)'$ were not needed in our arguments. However, in this section we will briefly describe a coordinatization of these planes.

Define the following products on K^2:

$$(\alpha,\beta)*(t,u)=(\alpha(u-t^2)+\beta t,\alpha(-\tfrac{1}{3}t^3+l(t))+\beta u)$$

$$(\alpha,\beta)\circ(t,u)=\begin{cases}(\beta t,\beta u) & \text{if } \alpha=0\\ (\alpha u+t^2+\alpha^{-1}\beta t,\alpha^{-1}(-\tfrac{1}{3}t^3+l(t))+\alpha^{-1}\beta(\beta u+t^2+\alpha^{-1}\beta t)) & \text{if } \alpha\neq 0.\end{cases}$$

Write $(a,b,c,d)=((a,b),(c,d))=[(a,c),(b,d)]$. Then the lines of $\mathscr{A}(l)$ are the set of points $(X,Y)\in(K^2)^2$ of the form $X=C$ or $Y=X*M+B$, while those of $\mathscr{A}(l)^*$ have the form $X=C$ or $Y=M*X+B$. The lines of $\mathscr{A}(l)'$ are the sets of points $[X,Y]$ of the form $X=C$ or $Y=M\circ X+C$.

The product $M\circ X$ has been normalized so that $(m,n)\circ(0,y)=(my,ny)$ for all m,n,y. Note that $M\circ(X+(0,y))=M\circ X+M\circ(0,y)$. It follows that the group T of q^3 translations of $\mathscr{A}(l)'$ consists of all mappings $[X,Y]\to[X,Y]+[(0,a),C]$. Such a translation has direction M if and only if $C=M\circ(0,a)$, in which case we will call it $\tau[M;a]$. Let $\tau(C):[X,Y]\to[X,Y]+[0,C]$, so $\tau(K^2)$ is the group of (∞',L'_∞)-elations.

Let $M\neq M'$. If $a+a'\neq 0$ then

$$\tau[M;a]\tau[M';a']=\tau[M+(a+a')^{-1}a'(M'-M);a+a'],$$

while $\tau[M;a]\tau[M';-a]=\tau(a(M-M'))$. Thus, $\langle\tau[M;K],\tau[M';K]\rangle$ contains the q groups $\tau[M+\lambda(M'-M);K]$, $\lambda\in K$, along with $\tau(K(M-M'))$. Consequently, *each orbit of this group consists of the q^2 points of a Baer subplane of $\mathscr{A}(l)'$.*

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