PRIMITIVE GROUPS HAVING TRANSITIVE SUBGROUPS OF SMALLER, PRIME POWER DEGREE[†]

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ABSTRACT

The groups in the title are classified, provided they are not too highly transitive.

Let G be a primitive permutation group on a finite set S of n points. In 1871, Jordan initiated the study of G under the additional assumption that there is a transitive subgroup H of degree m, where $1 < m < n$; that is, H fixes $n - m$ points and is transitive on the remaining points.

THEOREM 1. *If m is a prime power, and G is not $n - m + 1$-transitive, then G is one of the following groups in its usual 2-transitive representation: a collineation group of $PG(d-1,q)$ containing $PSL(d,q)$, where $d \geqq 3$; the full collineation group of $AG(d,2)$, where $d \geqq 3$; or a Mathieu group M_{22}, $\mathrm{Aut}(M_{22})$, M_{23}, or M_{24}.*

This result contains several recent theorems found in [2], [3] and [4].

PROOF. G is 2-transitive on S ([7, p. 32]). By [2, Sect. 6], we may assume that G is not 3-transitive. Let B be the complement of the given set of m points, so $|B| = k = n - m$. Let P be a Sylow subgroup of the pointwise stabilizer $G(B)$ of B, so that P is transitive on $S - B$. By [2, (3.6)], the distinct sets B^g, $g \in G$, form a design $\mathscr{D}$ whose lines have more than two points; moreover, if B is not a line of $\mathscr{D}$, then planes of $\mathscr{D}$ are well-defined and G is transitive on the set of planes.

Suppose first that B is a line of $\mathscr{D}$. Let $g \in G$ be such that $B \cap B^g$ is a point x. Since P is transitive on the lines $\neq B$ on x, the stabilizer P_1 of B^g in P has index

[†] This research was supported in part by NSF Grant GP 37982X.
Received January 1, 1974

$(n-k)/(k-1) < |S-B|$ in P. Hence, P_1 fixes no point of $S-B$. We may assume that P_1 normalizes P^g, and hence centralizes some $z \neq 1$ in $Z(P^g)$. Then $B^z = B$ as B is the set of fixed points of P_1. However, P^g is transitive on the lines $\neq B^g$ on x, so z must fix each line on x. Also, z fixes each point of B^g. Consequently, by a theorem of O'Nan [5], $\mathscr{D}$ consists of the points and lines of a projective space $PG(d-1,q)$, $d \geqq 3$, and G contains $PSL(d,q)$. Since $|S-B|$ is a prime power, $d = 3$.

Now suppose B is not a line of $\mathscr{D}$, and let E be any plane meeting B in a line. Then the sets $(E - B \cap E)^h$, $h \in P$, partition $S - B$, so $|E - B \cap E|$ is a prime power. By [2, (3.10)], the global stabilizer of E in G induces on E a group inheriting our hypotheses. Thus, each plane is a projective plane, so the points and lines of $\mathscr{D}$ form a projective space $PG(d-1,q)$ (see Veblen and Young [6]). Also, B is a subspace, and hence a hyperplane since $|S-B|$ is a prime power. Consequently, $G \geqq PSL(d,q)$.

By using a slightly more complicated argument (depending more heavily on [2]), we can generalize Theorem 1 as follows.

THEOREM 2. *Suppose G is a finite group primitive on a set S of n points. Let $B \subset S$, where $|B| = k < n-1$, and assume that G is not k-transitive. Then G is as in Theorem 1 if either of the following holds for the pointwise stabilizer $G(B)$ of B:*

(i) *$G(B)$ has a nilpotent Hall subgroup transitive on $S-B$; or*

(ii) *There is a prime $p \mid k-\mu$, where $\mu = \max\{|B \cap B^g| \mid B \neq B^g,\ g \in G\}$, and a Sylow p-subgroup P of $G(B)$, such that $C_{G(B)}(\Omega_1(Z(P))$ is transitive on $S-B$.*

Here, as usual, $\Omega_1(Z(P)) = \{g \in Z(P) \mid g^p = 1\}$.

We will only sketch the proof, which is similar to that of Theorem 1. We will assume familiarity with [2, Sect. 3]. Since $1 < k-\mu \mid n-k$ for Jordan groups, (i) is actually a special case of (ii). Thus, assume (ii). We may assume that G is not 3-transitive. Let $\mathscr{L}$ denote the (geometric) lattice of intersections of families of blocks.

Fix $F = B \cap C \in \mathscr{L}$ with B and C blocks and $|F| = \mu$. Let P be a Sylow p-subgroup of $G(B)$, and P_1 the stabilizer in P of C. Since $p \mid k-\mu$, P_1 fixes no point of $S-B$. Also, P_1 normalizes a Sylow p-subgroup Q of $G(C)$, and hence centralizes some $z \neq 1$ in $\Omega_1(Z(Q))$. Since $C_{G(C)}(z)$ is transitive on $S-C$, while z fixes B, it follows that z fixes all blocks containing F.

Choose $Y \in \mathscr{L}$ such that $Y \subseteq F$, z fixes all blocks containing F, and Y is minimal with respect to these conditions. Then $Y \neq \varnothing$, so we can choose $X \in \mathscr{L}$ maximal in Y (where possibly $X = \varnothing$). Note that $G(X)$ is 2-transitive on $S(X) = \{Y^g \mid X \subset Y^g, g \in G\}$. By [5], $S(X)$ is the set of points of a projective space, on which $G(X)$ induces at least the projective special linear group. Now the theorem follows from [2, Sect. 6].

Finally, we remark that it is very easy to use [5] to prove Theorem 2 if (i) and (ii) are replaced by the condition that $G(B)$ has an abelian subgroup transitive on $S - B$.

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