REFLECTIONS ON CONCRETE BUILDINGS*

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1. INTRODUCTION

In the last few years there have been many constructions and characterizations of finite groups acting chamber-transitively on finite building-like geometries. A number of these even dealt more generally with locally finite geometries: ones in which all stars are finite. Many of the examples, and some of the results, are concerned with quotients of affine buildings over locally compact local fields. The purpose of this note is two-fold: to discuss many of the known examples from a somewhat new point of view (§2), and to describe a characterization theorem due jointly to Liebler, Tits and myself (§3).

Consider an "algebraic" affine building Δ, defined by means of a simple algebraic group over a locally compact local field (see §2 for examples of concrete buildings of this sort). Let N be any discrete automorphism group of Δ having only finite many chamber-orbits. One can then form the quotient Δ/N. In general, this is merely a chamber-system [T3], not a simplicial complex, and hence does not correspond nicely to a geometry. However, by passing to a subgroup of N of sufficiently large finite index, it can always be arranged that Δ/N is a complex, and hence gives rise to a finite geometry whose diagram is the same as that of Δ (cf. [T2]). Evidently, this produces a wealth of finite geometries whose local structure is the some as that of Δ -- in fact, too many such geometries: classification is impossible. Moreover, Δ/N will usually inherit a very small group of automorphisms from Δ. This suggests that, if we require that Aut(Δ/N) be large, then it may be possible to study Δ/N more readily.

Consider the case in which G is a discrete group transitive on one of the types of vertices of Δ. In this situation, it is easy to see that Δ can be completely described as a

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coset geometry in terms of G and suitable subgroups (the stabilizers of the various simplexes in the star of some vertex). Now if N is a normal subgroup of G of sufficiently large finite index, then Δ/N will "be" a geometry, having the same diagram as Δ, and admitting an automorphism group G/N transitive on at least one type of vertices. Of course, the best situation of this sort, and the one of greatest group-theoretic interest, is that in which G is actually chamber-transitive on Δ: then G/N will be chamber-transitive on Δ/N as well. Nevertheless, even there it seems at present to be difficult to study Δ/N using standard geometric or representation-theoretic methods.

There is a sort of reverse direction from which to view this. Given a finite diagram geometry Δ whose rank 2 residues are buildings (i. e., a GAB [K4]), under mild restrictions (on B_3 subdiagrams) Tits has proved the important result that the universal cover $\tilde{\Delta}$ of Δ is a building [T3]. Moreover, if Δ has rank ≥ 4 and affine diagram then, by another remarkable result of Tits [T4], $\tilde{\Delta}$ arises from an affine building over a local field (all of which he had earlier classified: see [T1]), and $\Delta \cong \tilde{\Delta}/N$ for a discrete automorphism group N having finitely many orbits on Δ. Consequently, in the latter case we wind up back in the situation of the previous paragraphs.

With all of this in mind, we will construct a number of "sporadic" examples (Δ, G) consisting of an algebraic affine building Δ and an automorphism group G transitive on at least one of the types of vertices of Δ. In fact, in most of the cases discussed in §2 we will obtain a non-type-preserving group G transitive on more than one of the types of vertices, and hence which will have a type-preserving normal subgroup of index ≥ 2. However, the constructions make it clear that there is a definite advantage in not restricting our attention only to type-preserving automorphisms. These constructions are based on a simple observation concerning the action of reflections on certain algebraic affine buildings that clarifies some aspects of known examples. This approach was noticed in the course of studying somewhat analagous methods involving trees and their images: constructions of relevance to theoretical computer science [LPS].

The "reflection method" in §2 is intended to be simple and straightforward. In a future paper it will be shown how to generalize all of the examples in §2 (except for the existence of enough reflections to force transitivity) using entirely different methods (Strong Approximation for algebraic groups [Kne]). The present approach has the potential advantage of being more direct, and hence of making these particular buildings easier to visualize and hence easier to study.

2. REFLECTION CONSTRUCTIONS

In this section we will describe a mindless approach for constructing discrete automorphism groups of carefully chosen algebraic affine buildings that are transitive on (at least) two types of vertices. This method has implicitly been used for many of the known constructions of discrete chamber-transitive automorphism groups of such buildings.

As a typical example, consider the affine building Δ for $O(5,\mathbb{Q}_p)$ obtained from the vector space $V=\mathbb{Q}_p{}^5$ equipped with a nonsingular symmetric bilinear form (,) of Witt index 2. If $S\subseteq V$ let $L=\langle S\rangle_{\mathbb{Z}_p}$ denote the $\mathbb{Z}_p$-submodule spanned by S; if S contains a basis of V then L is called a lattice. Two lattices are regarded as equivalent if one is a scalar multiple of the other; [L] denotes the equivalence class containing the lattice L.

With this in mind, the vertices of Δ are the equivalence classes $[L^g]$, where $g\in\Omega(5,\mathbb{Q}_p)$ [the notation "$\Omega($, $)$" always meaning "$O($, $)'$"] and L is one of the following lattices:

$$L_0 = \langle e_1, \quad e_2, \quad f_1, f_2, u\rangle_{\mathbb{Z}_p}$$
$$L_1 = \langle e_1/p, e_2, \quad pf_1, f_2, u\rangle_{\mathbb{Z}_p}$$
$$L_2 = \langle e_1/p, e_2/p, f_1, f_2, u\rangle_{\mathbb{Z}_p}$$

where e_1, e_2, f_1, f_2, u is a basis such that $(e_i,f_i)= (f_i,e_i)=1$, $(u,u)=1$, and all other inner products are 0. Simplexes of Δ are images under $\Omega(5,\mathbb{Q}_p)$ of nonempty subsets of $\{[L_0], [L_1], [L_2]\}$.

If c is any nonsingular vector let r(c) denote the reflection

$$r(c) : v \longrightarrow v - 2(v,c)c/(c,c)$$

in $c^{\perp}$. Then $r(c) \in O(5,\mathbb{Q}_p)-\Omega(5,\mathbb{Q}_p)$. Moreover, $L_1=L_0^{r(c)}$ if $c=e_1+p\varepsilon f_1$ for any unit $\varepsilon \in \mathbb{Z}_p$ (e. g., $\varepsilon=1$; or $\varepsilon=\frac{1}{2}$ if $p>2$). Note that $(c,c)=2p\varepsilon$.

Conversely, if c is any element of L_0-pL_0 such that $(c,c)/2p$ is a unit in $\mathbb{Z}_p$, then one can show that $[L_0^{r(c)}]$ is a vertex of the same type as $[L_1]$, and is adjacent to $[L_0]$ (i. e., this pair of vertices forms an edge of Δ). Moreover, $p\{L_0+L_0^{r(c)}\}/pL_0$ is just the singular 1-space $\langle c+pL_0\rangle$ of the orthogonal space L_0/pL_0.

PROBLEM: Find subgroups G of $GO(5,\mathbb{Q}_p)$ [the group of all linear transformations preserving f projectively] that are transitive on the set of vertices of Δ of type 0 and 1, but such that the stabilizer of one of these vertices is finite. (Then G clearly will be discrete in the p-adic topology.)

The above remarks concerning reflections suggests a very simple strategy. For each singular 1-space of L_0/pL_0, try to find a suitable choice for $c \notin pL_0$ projecting onto the 1-space -- in which case the group generated by the corresponding reflections r(c) will be transitive on the set of vertices of type 0 and 1. Suitability is determined by the requirement that the above stabilizers are computable and finite. We will extend to $\mathbb{Q}_p^5$ a form f=(,) on $\mathbb{Q}^5$ that is positive definite, and then let G be the group $GO(\mathbb{Z}[1/p],f)$ of 5×5 matrices with respect to a suitably chosen basis β of $\mathbb{Q}^5$ that preserve the form projectively and have all entries in $\mathbb{Z}[1/p]$. We will arrange to have $L_0=\langle\beta\rangle_{\mathbb{Z}_p}$, so that the stabilizer of L_0 in G will consist of orthogonal matrices having entries in $\mathbb{Z}_p \cap \mathbb{Z}[1/p]=\mathbb{Z}$, and hence will be a finite group (since f is positive definite). Consequently, in each case $GO(\mathbb{Z}[1/p],f)$ will be discrete, while Δ will, in effect, be obtained entirely in terms of the rational space $\mathbb{Q}^5$.

Example 1. p=5, f=(,) is the usual scalar product, β is the standard basis. There are $(5+1)(5^2+1)$ singular 1-spaces in $L_0/5L_0$, represented by the images of the following 3 vectors under the group of all monomial matrices: (12000), (12120) and (11111). Each of the indicated vectors $c \in L_0$ satisfies $(c,c)=5$ or 10, so that $r(c) \in G$. Consequently, G behaves as required: transitively on 2 of the 3 types of vertices of Δ.

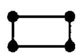
Example 1'. We could also replace f by the form $F_4 \oplus (1)$ that is the usual F_4-form on the first 4 coordinates and the usual scalar product on the fifth. While this form is rationally equivalent to the scalar product used in Example 1, matrices can now be written with respect a fundamental system (01-100), (001-10), (00010), $(\frac{1}{2}\text{-}\frac{1}{2}\text{-}\frac{1}{2}\text{-}\frac{1}{2}0)$, (00001). While the vector (00012) does not have integral inner product with respect to one of the basis vectors, the vector (0002-1) does, and the two vectors span the same 1-space mod 5. Consequently, we should be slightly more careful about which "reflection vectors" $c \in L_0$ are used, choosing the images of (13000), (00021), (12120), (01212), (11111), then we obtain another group transitive on 2 of the 3 types of vertices of Δ.

Example 2. $p=7$, f is the usual scalar product. There are $(7+1)(7^2+1)$ singular 1-spaces in $L_0/7L_0$, represented by the images of the following 3 vectors $c \in L_0$ under the group of all monomial matrices: (12300), (11120), (11222). Thus, G is again transitive on 2 types of vertices.

Example 3. $p=3$, f is the usual scalar product. This time there are $(3+1)(3^2+1)$ singular 1-spaces in $L_0/3L_0$, represented by the images of the single vector (11100) under the group 2^5S_5 of all monomial matrices. The latter group is, in fact, chamber-transitive on the O(5,3)-building. Thus, in this case G is actually chamber-transitive on Δ [Me]. (Moreover, there is also a chamber-transitive subgroup H of G that does not arise in the present reflection approach, in which the stabilizer of L_0 is 2^5F_{20}, where F_{20} is a Frobenius group of order 20 inside S_5 [Me]: see Remark 3 following Table 2.)

This exhausts all of the examples we have found using the reflection method with the usual scalar product in dimension 5. Clearly, one can move $[L_0]$ to its neighbors of type 1 using orthogonal transformations that are not reflections, and a suitably transitive group certainly need not have reflections doing this as well. Using entirely different methods (Strong Approximation) one can show that, if f is the usual scalar product on $\mathbb{Q}_p^5$ and if p>2, then $GO(\mathbb{Z}[1/p],f)$ is transitive on 2 of the 3 types of vertices of the $GO(5,\mathbb{Q}_p)$ building. (For the case p=2, see Example 7 below.) Note that this can be rephrased as the following factorization: $GO(\mathbb{Q},f)=GO(\mathbb{Q}\cap\mathbb{Z}_p,f)\cdot GO(\mathbb{Z}[1/p],f)$. There are similar stronger results concerning all of the other orthogonal groups we will consider; however, in this paper we merely want to examine constructions involving reflections.

We next turn to further examples of the above process involving larger-dimensional spaces.

Example 4. Let $f(u,v)=u\,\mathrm{diag}(111112)v^t$ on $V=\mathbb{Q}_3^6$. This turns V into an $O^+(6,\mathbb{Q}_3)$-space. The corresponding building Δ (with diagram ) is the 3-dimensional simplicial complex arising as above from the following 4 lattices:

$L_0 = \langle e_1, e_2, e_3, f_1, f_2, f_3\rangle_{\mathbb{Z}3}$

$L_1 = \langle e_1/3, e_2, e_3, f_1, f_2, f_3\rangle_{\mathbb{Z}3}$

$L_2 = \langle e_1/3, e_2/3, e_3/3, f_1, f_2, f_3\rangle_{\mathbb{Z}3}$

$L_3 = \langle e_1/3, e_2/3, f_3/3, f_1, f_2, e_3\rangle_{\mathbb{Z}3}$,

where e_1,e_2,e_3,f_1,f_2,f_3 is the usual type of basis. Once again let $G= GO(\mathbb{Z}[1/3],f)$. Since representatives of the $(3^2+3+1)(3+1)$ singular 1-spaces of $L_0/3L_0$ are provided by the vectors (111000), (100001) and (111101), we obtain transitivity on vertices of types 0 and 1, as above.

However, this time we can go further. There is an obvious projective orthogonal transformation $[e_i \to e_i/3; f_i \to f_i]$ sending L_0 to L_2; its determinant is 27. Similarly, if we send the standard basis of $\mathbb{Q}^6$ to (011-100), (-101100), (110100), (-1-1-1000), (000021), (0000-1-1), then the resulting transformation θ (of determinant 27) and its inverse both preserve f projectively and have all entries in $\mathbb{Z}[1/3]$. Thus, G is transitive on the set of all vertices of Δ! Moreover, $\langle\theta, r(111000)\rangle$ induces the full group D_8 of graph automorphisms of Δ.

Example 5. Let $f(u,v)=u\,\mathrm{diag}(122224)v^t$ on $V=\mathbb{Q}_3^6$. Then V has Witt index 2, and Δ has rank 3. This time, a slight refinement of our method is required. Namely, the 112 singular points in $L_0/3L_0$ are represented by the vectors c=(110000), (010001), (011100), (011111) and (2110001) up to monomial permutations preserving f. The last 2 of these vectors c satisfy f(c,c)=12. However, it is easy to check that they also satisfy $(c,L_0)\subseteq 2\mathbb{Z}$, so that 2(c,v)/(c,c) is in $(1/3)\mathbb{Z}$ for each $v\in L_0$ and hence $(L_0)^{r(c)}$ and L_0 are adjacent vertices of Δ.

Table 1 contains many vertex-transitive groups obtained by the reflection method described above. This table is by no means exhaustive: additional examples of this sort will undoubtedly be found by a closer examination of quadratic forms.

The columns of the table contain the following information. In column 1 we indicate either a row vector r such that the form is $(u,v)=u\,\mathrm{diag}(r)\,v^t$, or else the name of a root system. In the former case, all matrices are to be written with respect to the standard orthogonal basis; in the case of exceptional root systems or root systems of type A_n, a fundamental system of roots should be used. The desired group G is then the group of all matrices with respect to this basis that preserve the form projectively and have all entries in $\mathbb{Z}[1/p]$. Note that this is not a group of type-preserving transformations, but of course its largest type-preserving subgroup is transitive on at least 2 of the types of vertices of Δ.

Column 2 specifies the field over which the form is to be written. Columns 3 and 4 contain the name and diagram for the building Δ, following [T1]; the vertices 0 and 1 can be represented by any pair of nodes of the diagram interchanged by a graph automorphism. Each 0- or 1-vertex of Δ (i. e., vertex of type 0 or 1) corresponds to an equivalence class of $\mathbb{Z}_p$-lattices. The corresponding star in the building is the finite spherical building of the group in column 5; that group acts on L_0/pL_0, an $\mathbb{F}_p$-space inheriting a form from that of V.

Column 6 contains typical vectors $c \in L_0$ for use in §2. In general (cf. Example 5), (c,c) is p or 2p, and is 4 if p=2. Moreover, suitable permutations of the indicated vectors c, together with suitable sign changes, are further candidates for c. Note that c is always written with respect to the standard basis of the appropriate rational vector space, not with respect to a fundamental system of roots.

The last column contains the group induced on L_0, and hence (mod $\langle -1 \rangle$) the group induced projectively on the space L_0/pL_0 -- which is the same as the group induced by that stabilizer on the building Δ. Additional transitivity and references are also included when appropriate.

The exceptional root lattices are as follows:

E_8: $\{(x_i) \in \mathbb{Q}^8 \mid x_i + x_j \in \mathbb{Z}, \Sigma x_i \in 2\mathbb{Z}\}$

E_7: $x_7 = x_8$ in E_8

E_6: $x_6 = x_7 = x_8$ in E_8

F_4: $\mathbb{Z}^4 \oplus (\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})\mathbb{Z}$.

We will also need

A_n: $\{(x_i) \in \mathbb{Z}^{n+1} \mid \Sigma x_i = 0\}$.

In the Witt group [Cas] of $\mathbb{Q}_p$ for any p, the quadratic forms for root systems are as follows:

E_8: 0

E_7: 6(1) + (2) = (-2)

E_6: 5(1) + (3)

F_4: 4(1)

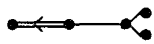
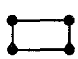
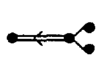
A_5: 3(1) + (2) + (3)

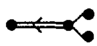
A_4: 3(1) + (5)

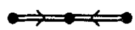
A_2: (1) + (3)

Finally, we note that "2" will be used to denote a cyclic group of order 2, since $\mathbb{Z}_2$ already stands for the ring of 2-adic integers.

TABLE 1. *Some discrete groups G transitive on at least 2 types of vertices.*
(In each case, G is gotten by using all matrices over $\mathbb{Z}[1/p]$
relative to a suitable basis for f's $\mathbb{Z}$-lattice and preserving f projectively.)

f	field	Δ	diagram	$star_0$	c	Transitivity, stabilizer, ref.
$E_8 \oplus A_2$	$\mathbb{Q}_2$	$^2D_5$		$O^-(10,2)$	(11110000000) (110000001-10)	$W(E_8)\times S_3\times 2$
$E_8 \oplus (2)$	$\mathbb{Q}_2$	B_4		$O(9,2)$	(111100000) (110000001)	$W(E_8)\times 2$
E_8	$\mathbb{Q}_2$	D_4		$O^+(8,2)$	(11110000)	chamber-tra. [K1] $W(E_8)$ tra. 4 vertex types
$E_6 \oplus A_2$	$\mathbb{Q}_2$	D_4		$O^+(8,2)$	(11110000000) (110000001-10)	$W(E_6)\times 2\times S_3\times 2$
E_7	$\mathbb{Q}_2$	B_3		$O(7,2)$	(11110000)	chamber-tra. [K1] $W(E_7)$
$E_6 \oplus (2)$	$\mathbb{Q}_2$	B_3		$O(7,2)$	(111100000) (110000001)	$W(E_6)\times 2\times 2$
E_6	$\mathbb{Q}_2$	$^2A'_3$		$O^-(6,2)$	(11110000)	chamber-tra. [MW] $W(E_6)\times 2$
A_6	$\mathbb{Q}_2$	A_3		$O^+(6,2)$	(11-1-1000)	chamber-tra. [K2] $S_7\times 2$ tra. all vertices
$A_4 \oplus A_2$	$\mathbb{Q}_2$	A_3		$O^+(6,2)$	(11-1-1000) (1-1001-10)	$S_5\times 2\times S_3\times 2$
A_5	$\mathbb{Q}_2$	B_2		$O(5,2)$	(11-1-100)	chamber-tra. [MW] $S_6\times 2$
$E_8 \oplus (2)$	$\mathbb{Q}_3$	B_4		$O(9,3)$	(211000000) (111100001)	$W(E_8)\times 2$
E_8	$\mathbb{Q}_3$	D_4		$O^+(8,3)$	(21100000)	$W(E_8)$ 3 chamber-orbits tra. 4 vertex types
$E_7 \oplus (2)$	$\mathbb{Q}_3$	D_4		$O^+(8,3)$	(211000000) (111100001)	$W(E_7)\times 2$
(11111111)	$\mathbb{Q}_3$	D_4		$O^+(8,3)$	(21100000) or (11100000) (11111100)	$2^8 S_8$
E_7	$\mathbb{Q}_3$	B_3		$O(7,3)$	(21100000)	$W(E_7)$

$E_6 \oplus (2)$	Q_3	B_3		$O(7,3)$	(211000000) (111100001)	$W(E_6)\times 2\times 2$
A_7	Q_3	B_3		$O(7,3)$	(-2110000) (111-1-1-100)	$S_7\times 2$
(1111111)	Q_3	B_3		$O(7,3)$	(1110000) or (211000) (1111110)	2^7S_7
(1111112)	Q_3	B_3		$O(7,3)$	(1110000) or (2110000) (1111110) (1000001) or (2000001) (1111001)	$2^6S_6\times 2$
(1222222)	Q_3	B_3		$O(7,3)$	(0111111) (1100000) or (2100000) (2111100)	$2^6S_6\times 2$
(111112)	Q_3	A_3		$O^+(6,3)$	(111000) or (211000) (100001) or (200001) (111101)	$2^5S_5\times 2$
(122222)	Q_3	A_3		$O^+(6,3)$	(011100) (110000) or (210000) (211110)	$2^5S_5\times 2$
(111111)	Q_3	${}^2A'_3$		$O^-(6,3)$	(111000) or (211000) (111111)	2^6S_6
(111122)	Q_3	${}^2A'_3$		$O^-(6,3)$	(111000) or (211000) (111110) (110011) (100010) or (200010)	$2^4S_4\times D_8$
(112222)	Q_3	${}^2A'_3$		$O^-(6,3)$	(101000) or (201000) (201111) (001110) (111100)	$2^4S_4\times D_8$
(122224)	Q_3	${}^2A'_3$		$O^-(6,3)$	(011100) (110000) or (210000) (010001) (011111) (211001)	$2^4S_4\times 2^2$

(11111)	$\mathbb{Q}_3$	B_2		O(5,3)	(11100)	chamber-tra. [Me] 2^5S_5
$F_4 \oplus (1)$	$\mathbb{Q}_3$	B_2		O(5,3)	(21100) (11001) or (11002)	$W(F_4)\times 2$
(11112) $F_4 \oplus (2)$	$\mathbb{Q}_3$	B_2		O(5,3)	(21100) (20001) (11111)	$2^4S_4\times 2$ $W(F_4)\times 2$
(12222)	$\mathbb{Q}_3$	B_2		O(5,3)	(11000) (21000) (01110) (21111)	$2^4S_4\times 2$
(11114)	$\mathbb{Q}_3$	B_2		O(5,3)	(11100) or (21100) (11001)	$2^4S_4\times 2$
(14444)	$\mathbb{Q}_3$	B_2		O(5,3)	(21100) (01110)	$2^4S_4\times 2$
(11122)	$\mathbb{Q}_3$	B_2		O(5,3)	(11100) or (21100) (11011) (10010) or (20010)	$2^3S_3\times D_8$
(11222)	$\mathbb{Q}_3$	B_2		O(5,3)	(10100) (11110) (00111)	$2^3S_3\times D_8$
(11144)	$\mathbb{Q}_3$	B_2		O(5,3)	(11100) or (21100) (20011) (01110)	$2^3S_3\times D_8$
(11444)	$\mathbb{Q}_3$	B_2		O(5,3)	(00111) (20110) (11100)	$2^3S_3\times D_8$
(11224)	$\mathbb{Q}_3$	B_2		O(5,3)	(10100) or (20100) (11001) (00101) (11110) (20111)	$D_8\times D_8\times 2$
(12244)	$\mathbb{Q}_3$	B_2		O(5,3)	(11000) or (21000) (01010) (21110) (20011) (01111)	$D_8\times D_8\times 2$

(12224)	$\mathbb{Q}_3$	B_2	•⇒•⇐•	O(5,3)	(11000) or (21000) (01001) (01110) (21101)	$2^3S_3\times 2^2$
(11111)	$\mathbb{Q}_5$	B_2	•⇒•⇐•	O(5,5)	(12000) or (13000) (11220) (11111)	2^5S_5
$F_4\oplus(1)$	$\mathbb{Q}_5$	B_2	•⇒•⇐•	O(5,5)	(13000) (00021) (11220) (01212) (11111)	$W(F_4)\times 2$
(11113)	$\mathbb{Q}_5$	B_2	•⇒•⇐•	O(5,5)	(12000) or (13000) (11220) (11001) (11121) (33301)	$2^4S_4\times 2$
(11111)	$\mathbb{Q}_7$	B_2	•⇒•⇐•	O(5,7)	(12300) (11120) (11222)	2^5S_5

Subbuildings. Many of the above examples over $\mathbb{Q}_p$, $p = 2$ or 3, are "contained in" (i. e., subbuildings of) the one arising from the E_8-form. In a sense, this is no accident: the subbuildings arise as fixed point sets of reflections, and each Weyl group associated with one of the indecomposable forms f has a unique class of reflections of a given "length".

For example, let Δ_8 be the E_8-example over $\mathbb{Q}_p$, p=2 or 3. If $u_1,\ldots,u_8$ is the standard basis for $\mathbb{Q}_8$ write $r_{i\pm j}=r(u_i \pm u_j)$ and $r=r_{7-8}$. If r fixes a 0- or 1-vertex [L] then it lies in the stabilizer of L in G, which is a Weyl group $W(E_8)$. The latter group has a unique conjugacy class of reflections. Consequently, $C_G(r)$ is transitive on the set of 0- and 1-vertices of Δ_8 fixed by r. The set of fixed chambers uniquely determines the corresponding subbuilding.

Similarly, other examples arise from sets of fixed chambers:

	forms over $\mathbb{Q}_2$
E_6	fixed chambers of $r_{6\text{-}7}$ on the E_7-example
A_5	fixed chambers of $r(\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})$ on the E_6-example
(1112)	fixed chambers of $r_{4\text{-}5}$ on the (11111)-example (cf. Table 2)
	forms over $\mathbb{Q}_3$
(11111111)	exact same 0- and 1-vertices as for the E_8-example
(1111111)	fixed 0- and 1-vertices of r_1 on the (11111111)-example
(1111112)	exact same 0- and 1-vertices as for the E_7-example
(111111)	fixed 0- and 1-vertices of r_1 on the (1111111)-example
(111112)	fixed 0- and 1-vertices of r_1 on the (1111112)-example
(11112)	fixed 0- and 1-vertices of r_1 on the (111112)-example
(11111)	fixed 0- and 1-vertices of r_1 on the (111111)-example

(Note that many of the other forms in Table 1 produce exactly the same 0- and 1-vertices [in fact, the same $\mathbb{Z}_p$-lattices] as those we have just considered.) The case of the A_6-example is somewhat different from the rest: there the building happens to arise as the set of fixed chambers of $\langle r_8, r(\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})\rangle$ on the E_8-example, but the reasoning used earlier to prove chamber-transitivity on this set of fixed chambers does not seem to apply.

Additional examples of this process can be found in Table 2 (see Remark 1 following that table).

Further notation: G_0 is the stabilizer of $[L_0]$, $G(2)$ consists of all matrices in G that are $\equiv 1 \pmod 2$, and hats denote projections mod $\langle -1\rangle$. Let Λ_0 be the $\mathbb{Z}$-lattice spanned by the basis indicated in the table (usually, a fundamental system).

PROPOSITION. (i) In each of the examples in Table 1 corresponding to the forms E_8 or E_7,

$$\hat{G} = \hat{G}(2) \rtimes \hat{G}_0,$$

so that $\hat{G}(2)$ acts regularly on the set of all vertices of type 0 or 1.

(ii) In the (11111)-example over $\mathbb{Q}_3$, $\hat{G}=\hat{G}(2)\cdot\hat{G}_0$ and $\hat{G}(2)\cap\hat{G}_0=2^4$.

PROOF. (i) Passage mod 2 sends $\langle\Lambda_0\rangle_{\mathbb{Z}[1/3]}$ to the orthogonal space of which $\hat{G}_0$ is the full orthogonal group. Thus, $\hat{G}/\hat{G}(2)\cong\hat{G}_0$, from which the conclusion follows.

(ii) All reflections in Table 1 have the form r(c) with (c,c)=3, so that $r(c)\equiv 1 \pmod 2$. Consequently, G(2) is transitive on the vertices of type 0 or 1. On the other hand, since $G_0=2^5S_5$ it is easy to see that $G(2)\cap G_0=2^5$. □

The regularity in part (i) implies that, in those cases, Δ can be described as a sort of Cayley graph.

<u>Problems:</u> 1. Does the conclusion of the Proposition hold in any other situations when p>2? Strong Approximation [Kne] greatly limits the possibilities here.

2. Does $\langle r(c) \mid c\in\Lambda_0, (c,c)=2p \text{ or } 2p/(2,p-1)\rangle$ equal the subgroup of G consisting of those transformations that preserve f (not merely projectively)? Since both groups are transitive on the vertices of type 0 and 1, this equality seems plausible.

Moreover, in some of the examples there are distinct choices c, c' such that $c-c'\in\Lambda_0$. In that case $r(c)r(c')\in G_0-\langle -1\rangle$.

3. In Table 1 we indicated which group G to use, as well as the stabilizers G_0 and G_1 of the vertices $[L_0]$ and $[L_1]$ (which are conjugate in G). Does $\langle G_0, G_1\rangle$ coincide with the subgroup G^+ of G consisting of all elements preserving the types 0 and 1? This is, in fact, the case when G acts sufficiently transitively -- in particular, when G is chamber-transitive. More precisely, if G_0 is transitive on the vertices of type 1 in the star of $[L_0]$ then $\langle G_0, G_1\rangle$ is transitive on the vertices of types 0 and 1, and has the same stabilizer G_0 as G, from which the desired equality follows. In which of the remaining cases is it also true that $G^+=\langle G_0, G_1\rangle$?

<u>Further examples</u>. Table 2 contains a list of the other known examples of transitive discrete groups that do not arise using the reflection method. When appropriate, we have again indicated an associated form, in which case G is defined exactly as in Table 1.

Stabilizers of vertices (i. e., lattices) corresponding to diagram end nodes are also included; in some instances these should have ⟨-1⟩ factored out in order to obtain an automorphism group of the building.

<u>TABLE 2</u>. *Further examples.*

	$\mathbb{Q}_2$	G_2		$G_2(2)$	chamber-tra. [K1] $2^3SL(3,2)$ (nonsplit), $G_2(2)$
	$\mathbb{Q}_3$	G_2		$G_2(3)$	tra. right hand vertex-type, stabilizer $G_2(2)$
(111111)	$\mathbb{Q}_2$	C-B_2		O(5,2)	chamber-tra. [K1] 2^6S_6
E_6	$\mathbb{Q}_3$	C-B_2		O(5,3)	chamber-tra. [Me] $W(E_6)\times 2$
(111113)	$\mathbb{Q}_3$	C-B_2		O(5,3)	chamber-tra. [KaMW] 2^6S_5
	$\mathbb{Q}_2$ $\mathbb{F}_2((t))$ $F_8((t))$	A_2	triangle	PSL(3,2) PSL(3,2) PSL(3,8)	chamber-tra. [KMW1,2;Me;Mu;T5] Frobenius groups: order 7·3 or 73·9
	$\mathbb{F}_q((t))$	A_2	triangle	PSL(3,q)	tra. all edges [T5] order $(q^2+q+1)\cdot 3e$ with $q=p^e$, p prime

<u>Remark 1</u>. Each of the first two examples in Table 2 arises as the set of fixed points of a triality automorphism on the corresponding E_8-example (see [K1] for the case of $\mathbb{Q}_2$). The E_6-example arises from the set of fixed 0-vertices of $r_{6\text{-}7}$ on the E_7-example. The (111113)-example arises from the set of fixed 0-vertices of $\langle r_{6\text{-}7}, r_{7\text{-}8}\rangle$ on the (11111111)-example.

<u>Remark 2</u>. Additional transitivity on one of these buildings Δ can be deduced once the group G is known to be transitive on 0-vertices. For example, the E_6- and (111113)-forms are rationally equivalent, and split over the ramified extension $\mathbb{Q}_3(\sqrt{-3})$ of

$\mathbb{Q}_3$. Consequently, in the notation of [T1], the corresponding building has type C-B_2 with diagram , and the star of each 0-vertex is isomorphic to the O(5,3)-building. Since G_0 induces a chamber-transitive group on that building, it follows that G is chamber-transitive on Δ in these cases.

The E_6- and (111113)-examples over $\mathbb{Q}_3$ are related in an even more concrete manner [KaMW]. First, note that the conclusion of the Proposition holds for the group $G=GO(f,\mathbb{Z}[1/3])$ of the E_6-example. (The proof is the same as for that Proposition.) Then $G/G(2) \cong GO(5,3)$, and the group for the (111113)-form contains as a subgroup of index 2 the preimage in G of the flag-transitive subgroup 2^5S_5 of GO(5,3) -- and is generated by that subgroup and a reflection.

The preimage in G of the flag-transitive subgroup 2^5F_{20} of 2^5S_5 is chamber-transitive on Δ. Once again, a reflection can be adjoined to obtain a slightly larger chamber-transitive automorphism group of Δ.

Remark 3. For a construction of the PSL(3,$\mathbb{Q}_2$)-example as the set of fixed chambers of an involutory automorphism of the A_6-example, see [K4].

Tree examples.

We conclude our discussion of concrete buildings with some having rank 2. Here, Δ is a tree.

Example 6. Let f be the usual scalar product on $V=\mathbb{Q}_2{}^5$. This time V has Witt index 1. The corresponding building Δ (of type ${}^2B_2$ in the notation of [T1]) arises by tensoring up to $\mathbb{Z}_2$ the two lattices $\Lambda_0=\mathbb{Z}^5$ and $\Lambda_1=(F_4$ root lattice$)\oplus\mathbb{Z}$, where the root lattice is spanned by $\mathbb{Z}^4$ on the first 4 coordinates together with the additional vector $c=(\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}0)$. Let $G=GO(f,\mathbb{Z}[\frac{1}{2}])$. Then $G_0=2^5S_5$ and $G_1=W(F_4)\times 2$, while $G_0\cap G_1=2^4S_4\times 2$. The usual scalar product is induced on $\Lambda_0/2\Lambda_0$; on the hyperplane H perpendicular to the vector $(11111)+2\Lambda_0$, the quadratic form $\frac{1}{2}(u+2\Lambda_0,u+2\Lambda_0)$ (mod 2) produces an $O^-(4,2)$ geometry (and $2\Lambda_1/2\Lambda_0$ is the singular 1-space $\langle 2c+2\Lambda_0\rangle$). Consequently, G_0 induces the full orthogonal group $O^-(4,2) \cong S_5$ on H, so that $G = GO(f, \mathbb{Z}[1/2])$ acts chamber-transitively on Δ [Me]. This example arises from the set of fixed points of r_1 on the (111111)-example.

Let $r = r(0001\text{-}1)$. The centralizer of r in G_i is $2^3 S_3 \times 2$, i=0, 1. The set of fixed chambers of r is the building Δ' (of type A_1) corresponding to the diagonal form (1112), and $C_G(r)$ acts chamber-transitively on Δ'. This situation was observed and studied in [We].

Example 7. Let f be the A_5-form over $\mathbb{Q}_3$. The resulting group is chamber-transitive on the building of type ${}^2B_2$, with stabilizers $S_6 \times 2$ and $(D_{12} wr 2) \times 2$ acting on the lattices

$$L_0 = \langle u_1-u_2,\ u_2-u_3,\ u_3-u_4,\ u_4-u_5,\ u_5-u_6 \rangle_{\mathbb{Z}3}$$

$$L_1 = \langle u_1-u_2,\ u_2-u_3,\ u_4-u_5,\ u_5-u_6,\ u_1+u_2+u_3-u_4-u_5-u_6 \rangle_{\mathbb{Z}3}$$

whose corresponding stars are, respectively, the $O^-(4,3)$ and $O(3,3)$ buildings.

This example arises by means of the set of fixed chambers of $r(\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})$ in its action on the E_6-example. The example was also found by T. Meixner.

In the next examples we will use the classical quaternion division algebra over the real numbers.

Example 8. Consider the group G of all quaternions $x = a\omega + bi + cj + dk$ with $a,b,c,d \in \mathbb{Z}[1/p]$ and $\pm x\bar{x}$ a power of the odd prime p, where $\omega = \frac{1}{2}(1+i+j+k)$. (Note that $\langle \omega, i, j, k \rangle_{\mathbb{Z}}$ is the ring of integral quaternions, which we have tensored up to $\mathbb{Z}[1/p]$. Then G is just the group of units in the resulting ring.) Since the algebra of rational quaternions splits when tensored up to $\mathbb{Q}_p$, we can view G as a group of 2×2 matrices over $\mathbb{Q}_p$ -- that is, as a subgroup of $GL(2,\mathbb{Q}_p)$. Then G acts transitively on the set of all vertices of the corresponding rank 2 building for $SL(2,\mathbb{Q}_p)$ [GP, p. 261], defined by the lattices

$L_0 = \langle u_1, \quad u_2\rangle_{\mathbb{Z}_p}$

$L_1 = \langle u_1/p, u_2\rangle_{\mathbb{Z}_p}$,

where u_1, u_2 is a basis of $V=\mathbb{Q}_p{}^2$.

This is a standard example in the theory of automorphic forms [GP]. Here $G_0 \cong SL(2,3)$. Note that G(2) is the kernel of the map $x \rightarrow x \pmod 2$, and $G/G(2) \cong PSL(2,3)$. If hats denote projections into $PGL(2,\mathbb{Q}_p)$, then $\hat{G}=\hat{G}(2)\rtimes\hat{G}_0$ and $\hat{G}(2)$ is regular on vertices (just as in our earlier Proposition). (Note that $\hat{G}(2)$ is a free group of rank $\frac{1}{2}(p+1)$ when $p\equiv 1 \pmod 4$ [GP, p. 263].)

Since G_0 induces A_4 on the star of $[L_0]$, G is chamber-transitive if p is 3, 5 or 11. In order to force further transitivity, adjoin the quaternion i+j, which normalizes G and whose square is the scalar -2. Then $G\langle i+j\rangle$ is also chamber-transitive when p is 7 or 23, since $G_0\langle i+j\rangle/\langle -1,2\rangle \cong S_4$.

Example 8'. Similar examples arise by letting the quaternion x range through a maximal order

$$\tfrac{1}{2}\langle 1+\tau'i+\tau j,\ \tau'i+j+\tau k,\ \tau i+\tau'j+k,\ i+\tau j+\tau'k\rangle_{\mathbb{Z}[\tau]}$$

($\tau^2-\tau-1=0$, $\tau'=1/\tau$) in the ring of quaternions over $\mathbb{Q}(\sqrt{5})$, so that $G_0 \cong SL(2,5)$ [Vi, p. 141]. Moreover, we obtain a chamber-transitive automorphism group of the $SL(2,\mathbb{Q}_p(\sqrt{5}))$-building when p is 3, 5, 19, 29, or 59.

Remark 4. Each of the above chamber-transitive groups automatically is an amalgam of the stabilizers of 2 adjacent vertices [Ser].

Remark 5. See [Ih] for a discussion of the subgroups of $SL(2,\mathbb{Q}_p)$, p odd, that act regularly on the vertices of the corresponding building.

3. CLASSIFICATION

It is not at all clear whether it is even reasonable to ask for a classification of the algebraic affine buildings Δ and discrete automorphism groups G that are transitive on at least one class of vertices. In view of [Ih], one should certainly assume that Δ has rank ≥3. Nevertheless, even in this case it can be shown that there are large numbers of examples of pairs (Δ, G) with G transitive on at least one of the types of vertices of Δ. On the other hand, while classification may be difficult or impossssible in general, it is possible under suitable additional hypotheses. This is indeed the case if chamber-transitivity is assumed:

THEOREM [KLT]. Let Δ be an algebraic affine building defined by a simple algebraic group $\mathcal{G}$ of relative rank ≥ 2 over a locally compact local field. Let G be a discrete, type-preserving, chamber-transitive group of automorphisms of Δ. Then $\mathcal{G}$ is one of the following:

a split group over $\mathbb{Q}_2$ of type A_2, B_2, G_2, A_3, B_3, or D_4;

a split group over $\mathbb{Q}_3$ of type B_2;

a nonsplit 6-dimensional orthogonal group splitting over $\mathbb{Q}_2(\sqrt{-1})$, $\mathbb{Q}_2(\sqrt{-3})$ or $\mathbb{Q}_3(\sqrt{-3})$; or

a split group of type A_2 over $\mathbb{F}_2((t))$ or $\mathbb{F}_8((t))$.

Moreover, in each case there are at most 6 conjugacy classes (in Aut Δ) of groups G, all of which have been explicitly determined.

All of the buildings Δ involved in the Theorem are contained in Tables 1 and 2 (the nonsplit orthogonal groups have types C-B_2, ${}^2A'_3$ and C-B_2, respectively, in Table 1 or 2). The groups G that arise are not quite the same as the ones in §2: this time G is type-preserving. Moreover, in some cases there are additional groups G, obtained either by restricting determinants of groups in Table 1 or 2 or by using the subgroup 2^4F_{20} of 2^4S_5 acting chamber-transitively on the O(5,3)-building. For a discussion of similar, purely group-theoretic results, see [Tim].

The proof of the Theorem involves several ingredients:

(i) The determination of all of the affine buildings Δ defined by algebraic groups $\mathcal{G}$ of relative rank ≥ 2 over a locally compact local field K [T1];

(ii) The classification of all chamber-transitive automorphism groups of finite spherical buildings of rank ≥ 2 [Sei];

(iii) Character degree bounds [LS; FT];

(iv) Geometric properties of Δ [BrT];

(v) Properties of "good" unipotent elements (cf. [BoT; T6]); and

(vi) A theorem [Ve] stating that $\mathcal{G}(K)$ cannot have a discrete cocompact subgroup if char $K = p > 2$ except when $\mathcal{G}$ has diagram $\tilde{A}_{r-1}$. (Due to the restriction on p, this result is not actually used in [KLT].)

Nevertheless, since there are 12 buildings and many more (conjugacy classes of) groups to be characterized, the arguments eventually degenerate into a case analysis of a small number of buildings and possible groups. There is also a separation into the very different situations in which the field K has characteristic 0 or characteristic $p \neq 0$. Fortunately, in any characteristic, each finite subgroup of Aut K is solvable. This almost always reduces considerations to a linear group rather than a semilinear one.

In the remainder of this paper we will give an idea of the types of arguments used by considering some specific cases.

<u>Notation.</u>

G, $\mathcal{G}(K)$, Δ as above.

C: a chamber of Δ, with vertices $1,\dots,r$ ($r \geq 3$).

G_i: stabilizer of i in G.

G_{ij}: stabilizer of i and j in G.

Star(i): star of i.

K_i: the kernel of the action of G_i on Star(i).

Example: Δ of type D_4. The diagram is  with end nodes 1,2,3,4. By [Sei], $G_1/K_1 \trianglerighteq P\Omega^+(8,q)$ for the appropriate power q of p. It follows that the last term $(G_1)^{(\infty)}$ in the derived series of G_1 acts chamber-transitively on Star(1). Then $\langle (G_1)^{(\infty)}, (G_2)^{(\infty)} \rangle$ acts chamber-transitively on Δ; we will assume that it coincides with G. On the other hand, G_1 is a finite linear group acting projectively on the 8-dimensional vector space V underlying $\mathcal{G}(K) = P\Omega^+(8,K)$. Since G is a perfect group it lies in $P\Omega(V)$.

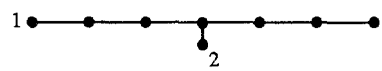
Case char K=0. Here G_1 is a finite linear group acting projectively on the 8-dimensional vector space V of characteristic 0. By [LS], q=2, $G_1 \cong \Omega^+(8,2)$, and the representation of G_1 on V must arise from $W(E_8)$. Then G_{12} is a parabolic subgoup of G_1 of the form $2^6\Omega^+(6,2) \cong 2^6A_8$. There are 3 classes of parabolics of G_1 of this sort, one of which arises from a monomial group within $W(E_8)$. It is easy to see that that 2^6A_8 lies in exactly 2 different subgroups $W(E_8)$ of $P\Omega^+(8,K)$ -- with one obtained from the other by means of a reflection r normalizing our 2^6A_8 (and generating a monomial group 2^7A_8 with it), just as in §2. Then $G=\langle G_1, r\rangle$ is uniquely determined up to conjugacy in Aut $\mathcal{G}(K)$.

Case char $K=p\neq 0$. This time $(G_1)^{(\infty)} \cong P\Omega^+(8,q)$ acts (projectively) on V in the natural manner. Similarly, the representations of $(G_i)^{(\infty)}$ and hence also of G_i on V are the usual ones (i=1,2,3,4). Let U be a Sylow p-subgroup of G_C. Using G_{1i} (i=2,3,4) we see that U fixes exactly 3 totally singular subspaces of V whose stabilizers in $\mathcal{G}(K)$ have $P\Omega^+(6,q)$ as a section (subspaces of dimensions 1, 4 and 4). These are just the subspaces fixed by G_{1i} (i=2,3,4). Permuting the subscripts and noting that $G_1=\langle G_{12}, G_{13}\rangle$, we see that $G_{12}=G_{34}$. Viewed within Star(1), this states that G_{12} fixes the additional vertices 3 and 4, which is ridiculous.

The general case of the Theorem follows a somewhat similar pattern, at least when K has characteristic 0. In that case, G_1 and K_1 are greatly restricted: bounds on the degrees of representations of G_1/K_1, together with the representation produced by the embedding $G < \text{Aut}\ \mathcal{G}(K)$, usually force K_1 to be 1 and always force G_1/K_1 to be one of a very small list of examples in small dimension and characteristic. The representation of G_1 can then be determined, and is usually unique up to conjugacy in Aut $\mathcal{G}(K)$. The same analysis can then be used for a suitably chosen second vertex 2, and the groups G_1 and G_2 can be played off against one another (using, among other things, the fact that G_{12} projects onto a parabolic subgroup of both G_1 and G_2 -- unless G_i/K_i (i=1 or 2) is one of the exceptional chamber-transitive groups in [Sei]).

On the other hand, when char K=p the approach used for D_4 applies to the diagram $\tilde{A}_{r-1}$ but not to many others. We refer to [KLT] for an approach using the geometry of Δ (i. e., using the metric on the topological space underlying Δ [BrT]). The result in [Ve] could be used when p>2. However, the most elementary approach is as in the following example.

<u>Case</u> E_7, char K=p. Label the diagram



By [Sei], G_1/K_1 contains $E_7(q)$. Since G acts projectively on the 56-dimensional module V for $\mathcal{G}(K)$ over K, it follows that $K_1=1$. Also, G_2/K_2 contains PSL(8,q). We may assume that G is perfect and hence lies in SL(V). The groups G_1, G_2 and G_{12} have a common Sylow p-subgroup U. Calculating $|U|$ in G_1 and G_2, we find that $|U \cap K_2| = q^{27}$. However, a straighforward root system calculation in $E_7(q)$ shows that the parabolic subgroup G_{12} of $E_7(q)$ has no normal subgroup of order q^{27}.

A similar approach works in almost all cases, but must be tailored to each possible type of building Δ. The only significantly different situations are those of buildings of type A_2 in which G_1 is a Frobenius group of order 7·3 or 73·9; see [T8] for an indication of the approach used in these cases.

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