STRONGLY REGULAR GRAPHS FOR $G_2(q)$

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Abstract. The points of each generalized hexagon with $s = t$ are the vertices of a strongly regular graph.

In [DK] the authors constructed symmetric designs from generalized hexagons with $s = t$. The construction amounted to merging associated distance relations. In [DK] it was observed that these designs have null polarities, which implicitly means that there is an associated strongly regular graph.

Let X be the set of points of a generalized hexagon $\mathscr{H}$ for which $s = t$. Let d denote distance in the point graph. For points x and y write $x \sim y \iff d(x, y)$ is 1 or 2. This defines a graph $\Gamma(\mathscr{H})$ with vertex set X.

Theorem 0.1. *The graph $\Gamma(\mathscr{H})$ is strongly regular with $(s^6 - 1)/(s - 1)$ vertices and valence $s(s^4 - 1)/(s - 1)$.*

This is implicit in [DK]. When s is a prime power this graph has the same parameters as the singular points graph of $\Omega(7, q)$. The sets $x^\perp = \{y \mid d(x, y) \le 2\}$, $x \in X$, are the blocks of the symmetric design in [DK]. It follows that $\mathrm{Aut}\,\Gamma(\mathscr{H})$ was determined in [DK] when $\Gamma(\mathscr{H})$ is the $G_2(q)$ general hexagon or its dual. In particular, the graph for the dual hexagon does not appear to be isomorphic to any strongly regular graph in print if q is not a power of 3 (so that the $G_2(q)$ generalized hexagon is not self-dual).

The graph $\Gamma(\mathscr{H})$ is interesting only because it is a strongly regular graph associated with the group $G_2(q)$, where that group has rank 4 on the vertices.

References

[DK] U. Dempwolff and W. M. Kantor, Symmetric designs from the $G_2(q)$ generalized hexagons. J. Comb. Theory (A) 98 (2002) 410–415.

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