FINITE SIMPLE GROUPS VIA p-ADIC GROUPS

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In this note we will describe a recent approach to the study of the following situation.

(*) G is a finite group generated by a family $\{P_1,\ldots,P_r\}$ of subgroups, $r \geq 3$, such that the following hold for some prime p and all i, j for which $1 \leq i < j \leq r$:

(1) $O^{p'}(P_i/O_p(P_i))$ is a rank 1 Chevalley group of characteristic p;

(2) $B = \bigcap P_i$ projects onto a Borel subgroup of each group in (1); and

(3) $O^{p'}(\langle P_i,P_j\rangle/O_p(\langle P_i,P_j\rangle))$ is either a rank 2 Chevalley group of characteristic p, or the product of the projections of $O^{p'}(P_i)$ and $O^{p'}(P_j)$.

The main theorem concerning (*) is as follows.

Theorem 1 (Niles [9]). Assume (*) together with

(i) No Chevalley group in (1) is $A_1(2)$, $A_1(3)$, ${}^2A_2(4)$, ${}^2B_2(2)$ or ${}^2G_2(3)$,

(ii) No group in (3) is $A_2(4)$, and

(iii) Each product arising in (3) is a direct product.

Then $\langle O^{p'}(P_i) \mid 1 \leq i \leq r\rangle$ is a normal subgroup of G having a rank r BN-pair.

Clearly, the results of Tits [15] then determine G modulo $K = \bigcap\{B^g \mid g \in G\}$, if the associated Dynkin diagram is connected.

While the assumptions in (*) are natural, the additional ones in Theorem 1 are unfortunate, and even annoying. However, there are examples showing that the groups P_i in (*) do not necessarily produce BN-pairs unless some kind of additional assumption is made. Before describing these examples, we will need additional notation.

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The diagram of (*) is the obvious analogue of the Dynkin diagram of a group with a BN-pair. Namely, its nodes can be identified with the groups P_i, and two nodes P_i, P_j are joined by 0, 1, 2, 3 or 4 edges when the group in (3) corresponds to a rank 2 BN-pair with Weyl group of order 4, 6, 8, 12 or 16, respectively. The groups P_i are regarded as "minimal parabolic subgroups", while the r groups $M_i = \langle P_j \mid j \neq i \rangle$, $1 \leq i \leq r$, are "maximal parabolic subgroups".

The following Table contains essentially all of the examples of (*), other than those arising from BN-pairs, presently in print or at least in preprints. (There are even more examples that have been announced, some due to myself but most due to Köhler, Meixner and Wester. Here, "essentially" refers to the omission of some extensions and homomorphic images. Examples of this are groups $\Omega^{\pm}(6,m)$ and $\Omega(5,7)$ in row 8.) The Table only contains the groups M_i; in each case, the groups P_i can be determined using (2). When several M_i are isomorphic, only one of them is listed.

In row 1, the middle group M_i has $M_i/O_5(M_i) = 2(PSL(2,5)PSL(2,5))4 \cong 2A_6 4$. This shows that hypothesis (iii) of Theorem 1 is essential. On the other hand, the remaining examples in the Table show that some of the groups $P_i/O_p(P_i)$ can be $PSL(2,2)$ or $PSL(2,3)$ without $\{P_1,\ldots,P_r\}$ arising from a BN-pair: at least part of hypothesis (i) is essential.

The first 5 columns of the Table are self-explanatory. The last column will be discussed later. The next-to-last column is headed by an "m", which is either definitely or seemingly irrelevant (denoted "-") or stands for all the indicated primes (e.g., all except 2 in the second row). In the latter case, each m produces an example: there is then an infinite family of examples, and for all but the first few values m does not divide any order $|M_i|$. Thus, Niles' theorem fails dramatically when there is a parameter m. It should be noted that m actually does not have to be a prime in any of these cases; for example, in the second row m can be any odd integer > 1, and G then is $P\Omega^+(8,\mathbf{Z}/(m))$, which is far from simple when m is composite.

The references in the Table provide constructions. We next turn to the problem of classification. Here, the main results are due to Timmesfeld. Before stating them, we will add one further assumption to (*):

4) The diagram of $\{P_1,\ldots,P_r\}$ is connected.

TABLE

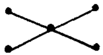
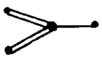
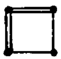
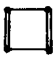
Diagram	G	p	Ref.	Maximal parabolics	m	Universal 2-cover
	LyS	5	[5]	$G_2(5), 5^{1+4}2A_6 4, 5^3SL(3,5)$	-	?
	$P\Omega^+(8,m)$	2	[6] [2] if m = 3	$\Omega^+(8,2), SL(2,3)^4 2^3$	$\neq 2$	$P\Omega^+(8,\mathbb{Q}_2)$
	$\Omega(7,m)$	2	"	$2^6A_7, Sp(6,2), (SL(2,3)^2\times A_4)2^2$	$\neq 2$	$P\Omega(7,\mathbb{Q}_2)$
	$\Omega(7,m)$	2	"	$2^6A_7, Sp(6,2), P\Omega^-(6,3)2$	$\neq 2$	??
	A_7	2	[8,2]	$A_6, (A_4\times 3)2, PSL(3,2)$	-	Itself
	$P\Omega^\pm(6,m)$	2	[6] [11,5,2] if m = 3	$2^4A_6, SL(2,3)^2 2^2$	$\neq 2$	$P\Omega(\mathbb{Q}_2,f)$
	$G_2(m)$	2	[6] [4] if m = 3	$G_2(2), SL(2,3)^2 2, 2^3SL(3,2)$	$\neq 2$	$G_2(\mathbb{Q}_2)$
	$P\Omega^\pm(6,m)$ $7^5\Omega(5,7)$ if m = 7	2	[7] [2] if m = 3	A_7	$\neq 2$	$P\Omega(\mathbb{Q}_2,f)$

TABLE (contd.)

Diagram	G	p	Ref.	Maximal parabolics	m	Universal 2-cover
	PSU(6,m)	3	[7] [5] if m = 2	$P\Omega^-(6,3)3^{1+4}2^{1+4}S_3$	≠ 3	?
	PSU(5,m) $2^{1+8}SU(4,2)$ if m = 2	3	[7]	$\Omega(5,3),3^{1+2}2^{1+4}3$	≠ 3	?
	Suz	2	[11]	$2^4 2^6 3A_6, 2^2 2^8(S_3\times A_5), PSU(4,2)$	–	?
	$\Omega^+(8,2)$	3	[5]	$(3\times PSp(4,3))2, 3^4 2^3 S_4$	–	?
	$G_2(2)$	2	[2]	$G_2(2), SL(2,3)^2 2$	–	?
	M^c	2	[12]	$2^4 A_7, 2A_8, PSU(4,3)$	–	??
	PSU(3,5)	2	[8,6]	A_7	–	??

Theorem 2 (Timmesfeld [14]). If $r = 3$ then the diagram has a pair of nonadjacent vertices.

Theorem 3 (Timmesfeld [13]). If the diagram has no multiple edges, then either $\{P_1,\ldots,P_r\}$ arises from a BN-pair, or else the diagram is

 or  and the M_i are exactly as in the Table.

The remainder of this note is concerned with the problem of going a bit further than Theorem 3.

Theorem 4. In Theorem 3, if G is simple then $G \cong P\Omega^+(8,m)$, $\Omega(5,7)$, or $P\Omega^{\pm}(6,m)$ for some odd prime m. (Here, if $m \neq 7$ then $\Omega^+(6,m)$ occurs if and only if $m \equiv 1$, 2, or 4 (mod 7).)

However, we will only outline the proof. On the other hand it will be clear that the ideas involved go far beyond Theorem 4.

Assume (*), and consider the set G/B. The groups P_i determine natural equivalence relations on G/B (namely, $Bg \underset{i}{\equiv} Bh \Longleftrightarrow P_ig = P_ih$). These turn G/B into a chamber system Δ (Tits [16]). In [16] it is shown that there is a universal 2-cover $\tilde{\Delta}$ of this chamber system, arising from a group $\tilde{G}$ generated by a family $\{\tilde{P}_i,\ldots,\tilde{P}_r\}$ of subgroups, such that there is an epimorphism $\sigma:\tilde{G} \to G$ mapping $\tilde{P}_i$ isomorphically onto P_i (with a similar statement for the groups $\langle\tilde{P}_i,\tilde{P}_j\rangle$). Moreover, the map σ has a standard type of universal property. Tits [16] then goes on to show that the chamber system $\tilde{\Delta}$ produced by $\tilde{G}$ and the $\tilde{P}_i$ "is" a building, provided that each subdiagram •——⇐• of the chamber system G/B arises from a rank 3 Chevalley group.

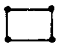
Digression. The fourth, fifth, and last two rows of the Table contain examples in which subdiagrams •——⇐• do not arise from Chevalley groups. The corresponding chamber systems $\tilde{\Delta}$ cannot arise from buildings; and in the fifth row, it is known that $\tilde{\Delta} = \Delta$ (due to Ronan).

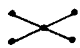
Now consider the case where $\tilde{\Delta}$ is a building and the diagram of Δ (and $\tilde{\Delta}$) is an extended Dynkin diagram. In this case, the building $\tilde{\Delta}$ is an affine building. If $r \geq 4$, then Tits [17] has classified all of these: they arise (via [3]) from groups over complete local fields (including the possibility of skew fields). Since $\tilde{G} \leq \mathrm{Aut}\,\tilde{\Delta}$, and the latter group is $\mathrm{Aut}\,G^*$ for a suitable algebraic group G^*, more information can be obtained concerning $\tilde{G}$.

Consider the case  of Theorem 3. Here, all of Tits' results apply, and show that $\tilde{\Delta}$ arises from the affine building associated to a group $D_4(F)$ for a complete local field F. Moreover, since $\sigma:\langle\tilde{P}_i,\tilde{P}_j,\tilde{P}_k\rangle \to \langle P_i,P_j,P_k\rangle$ is a cover, it is an isomorphism. Thus, each M_i is isomorphic to a group $\tilde{M}_i = \langle\tilde{P}_j \mid j \neq i\rangle \leq \tilde{G} \leq \mathrm{Aut}\, D_4(F)$. Moreover, the residue field of F must be $GF(2)$.

Since we know that $\tilde{M}_i \cong \Omega^+(8,2)$ for 4 values of i, this places restrictions on F. In fact, it is easy to show that F has characteristic 0, and that $\tilde{G} \leq D_4(F)$. It is only slightly harder to use two groups $\tilde{M}_i$ to show that (with respect to a suitable basis) $\tilde{G} \leq D_4(\mathbb{Q}_2)$, and then even that $\tilde{G} = P\Omega(\mathbb{Z}[1/2], \sum_1^8 x_i^2)$. This is, in fact, the situation encountered in [6].

Finally, G must be a finite simple homomorphic image of $\tilde{G}$. In the course of his work on the Congruence Subgroup Problem, Prasad [10] has shown that the only such finite images are $P\Omega^+(8,m)$ with m prime.

The case  is similar. In fact, Theorem 3 can be proved in this manner when $r \geq 4$ by first reducing to the case of extended Dynkin diagrams. (The case $r = 3$ is fairly easy.) Moreover, similar results can be proved for the case of all extended Dynkin diagrams of rank $r \geq 4$, provided that all B_3 subdiagrams arise from buildings. In a rather different direction, Aschbacher [1] has classified all situations (*) with $K = 1$ whose diagrams are diagrams of spherical buildings: he showed that only BN-pairs and the A_7 example (row 5) can arise.

There are many open problems, some of which are implicit in the above discussion. Others concern the last column of the Table. That column involves the identification of the universal 2-cover of the example or family of examples. In some cases, this cover is the building $\tilde{\Delta}$ associated with a 2-adic group. The corresponding group $\tilde{G}$ is then flag-transitive on $\tilde{\Delta}$, but is much smaller than $\mathrm{Aut}\, \tilde{\Delta}$ (just as in the  situation discussed earlier). However, usually the universal 2-cover is not known, in which case there is a question mark. Two question marks indicate that the universal 2-cover is not a building (in all other cases it is a building).

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