JOURNAL OF ALGEBRA 62, 232–234 (1980)

Linear Groups Containing a Singer Cycle*

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Communicated by Marshall Hall, Jr.

Received January 16, 1979

A Singer cycle of $GL(n, q)$ is an element of order $q^n - 1$. In this note the following result will be proved:

THEOREM. *If G is a subgroup of $GL(n, q)$ containing a Singer cycle, then $G \trianglerighteq GL(n/s, q^s)$ for some s, embedded naturally in $GL(n, q)$.*

If G induces a primitive permutation group on the set X of points (1-spaces) of the underlying vector space V, then well-known results of Burnside [1, p. 341] and Schur [6] imply that G acts on X 2-transitively or as a regular or Frobenius group of prime degree; the theorem then follows from [2; 5]. We must thus take a nontrivial block of imprimitivity Δ for G on X, and analyze the action of G on Δ^G. It should be noted that the proof is elementary in the same sense as in [2]: no purely group theoretic classification theorems are required.

Proof. We will employ induction on n. If $n = 2$, use of Dickson [4, Ch. 12] readily yields the result, so suppose that $n \geqslant 3$.

We may assume that G is imprimitive on X. Choose Δ as above with $|\Delta|$ minimal, and let K denote the kernel of the action of G on Δ^G. Let $A < G$ be generated by a Singer cycle, and set $B = K \cap A$. Then B^Δ is transitive, and B contains the group S of scalar transformations of V.

Clearly, $\mathrm{Hom}_A(V, V) = A \cup \{0\} \cong GF(q^n)$. The additive subgroup $\langle B \rangle$ of $\mathrm{Hom}_A(V, V)$ is closed under addition and multiplication, contains S, and hence is $GF(q^s)$ for some $s \mid n$. If $\delta \in \Delta$, then $\delta^{\langle B \rangle}$ consists of 0 and the set of points in the subspace $\langle \Delta \rangle$ of V spanned by Δ.

Suppose first that $\langle \Delta \rangle \neq V$, so $s < n$. Note that $\langle \Delta \rangle^G = \langle \Delta \rangle^A$ (since for each $g \in G$ there is an $a \in A$ such that $\langle \Delta \rangle^g = \langle \Delta^g \rangle = \langle \Delta^a \rangle = \langle \Delta \rangle^a$). But $\langle \Delta \rangle - \{0\}$ is an orbit of $\langle B \rangle - \{0\}$. Thus, $\langle \Delta \rangle^G$ can be identified with the set of points of $PG(n/s - 1, q^s)$. Since G permutes these points, if $n > 2s$

* This research was supported in part by the Science Research Council and the National Science Foundation.

0021-8693/80/010232–03$02.00/0

then $G \leqslant \Gamma L(n/s, q^s)$ by the Fundamental Theorem of Projective Geometry, and we may apply induction as $s > 1$. Suppose that $n = 2s$, and let $\mathfrak{A}$ consist of the vectors in V and the set of all cosets of members of $\langle \Delta \rangle^G$. Then $\mathfrak{A}$ is the affine plane $AG(2, q^s)$, whose collineation group is $V \cdot \Gamma L(2, q^s)$ (cf. Dembowski [3, pp. 31–32, 131–132]). Thus, $G \leqslant \Gamma L(n/s, q^s)$ once again.

We may thus assume that $\langle \Delta \rangle = V$ and $\langle B \rangle = \mathrm{Hom}_A(V, V)$. Now $N_{\Gamma L(n,q)}(B)$ acts on $\mathrm{Hom}_A(V, V)$, and hence is $\Gamma L(1, q^n)$. We may thus also assume that B is not normal in G.

After these reductions, we will aim for a contradiction. Clearly, $G_\Delta{}^\Delta$ is primitive, and $G_\Delta{}^\Delta \trianglerighteq K^\Delta \geqslant B^\Delta$ with B^Δ cyclic and transitive. Note that K^Δ is also primitive. (For, each complete system of blocks of imprimitivity for K^Δ consists of all orbits of a subgroup of the cyclic group B^Δ. Since $G_\Delta{}^\Delta$ permutes these systems, it must fix each system, and hence each system for K^Δ is also one for $G_\Delta{}^\Delta$.)

The theorems of Burnside and Schur cited above thus leave us with two cases to consider: (i) K^Δ is 2-transitive, and (ii) $B^\Delta \trianglelefteq G_\Delta{}^\Delta$.

(i) Let L be a line of $PG(n, q)$ such that $|\Delta \cap L| = l \geqslant 2$. Then l is independent of the choice of both Δ in Δ^G and of L. Clearly, $l < q + 1$, as otherwise Δ would be a subspace. Let $x \in \Delta \cap L$ and $y \in L - \Delta$, where $y \in \Delta' \in \Delta^G$. Then $y^{K_x} \subseteq \Delta'$ and $|\bigcup\{L^k \cap \Delta' \mid k \in K_x\}|$ is either $(\delta - 1)/(l - 1)$ or $l \cdot (\delta - 1)/(l - 1)$, where $\delta = |\Delta| > l$ (since $V = \langle \Delta \rangle$ has dimension $n \geqslant 3$). But $l(\delta - 1)/(l - 1) > \delta$, so we must have $|L \cap \Delta'| = 1$.

Now K_L fixes $L - \Delta$ pointwise, while $K_L^{\Delta \cap L}$ is 2-transitive and $K_L{}^L \leqslant PGL(2, q)$. Consequently, $l = q$ except perhaps if $l = 2$ and $q = 3$.

Suppose that $l = q$. Let E be a plane containing three noncollinear points of Δ. If a line contains two points of $\Delta \cap E$ then it contains exactly q points of $\Delta \cap E$ and 1 of $\Delta' \cap E$. Thus, if $q > 2$ then $\Delta \cap E$ is an affine plane with line at infinity $\Delta' \cap E$. (A line of E containing two points of $E - \Delta$ can contain at most one point of $\Delta \cap E$.) If $q = 2$ then $\Delta \cap E$ may be a triangle $\{x_1, x_2, x_3\}$, and the third points on $\langle x_1, x_2 \rangle$, $\langle x_2, x_3 \rangle$ and $\langle x_3, x_1 \rangle$ are collinear. In either case, $\Delta' \cap E$ contains a line, which is impossible.

This leaves the possibility $l = 2$ and $q = 3$. Here, $|y^{K_x}| = \delta - 1$, so K_x fixes a unique point $x' \in \Delta'$. Clearly, L meets a third member $\Delta'' \neq \Delta, \Delta'$ of Δ^G, and K_x also fixes some $x'' \in \Delta''$. Then $K_x = K_{x'} = K_{x''}$. However, there are $\frac{1}{2}\delta(\delta - 1)/\delta$ lines on x' meeting Δ twice, and these yield a $K_{x'}$-invariant set of $\frac{1}{2}(\delta - 1)$ points of Δ''. Thus, $\frac{1}{2}(\delta - 1)$ is $\delta - 1$ or 1. Now $\delta = 3$, so $n \leqslant 3$ and $G \leqslant GL(3, 3)$. But here $|X| = 13$ contradicts the imprimitivity of G.

(ii) Since B is not normal in G, the pointwise stabilizer $K(\Delta)$ of Δ in K must contain S properly. Since $\langle \Delta \rangle = V$ the group $K(\Delta)$ must be diagonalizable. Now all point-orbits of the monomial group $K(\Delta)B$ have length δ, which is ridiculous.

This contradiction completes the proof of the theorem.

Remark. It would be desirable to have an equally elementary determination of all subgroups of $GL(n, q)$ containing $A \cap SL(n, q)$.

References

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