Math. Z. 192, 45-50 (1986)

Mathematische Zeitschrift

Some Generalized Quadrangles with Parameters q^2, q

William M. Kantor*

Dept. of Math., Univ. of Oregon, Eugene, Oregon 97403, USA

To Professor Helmut Wielandt, to commemorate his seventy-fifth birthday

1. Introduction

In [2] a new generalized quadrangle with parameters q^2, q was constructed using the $G_2(q)$ generalized hexagon ($q \equiv 2 \pmod 3$, $q > 2$). In addition, an elementary group theoretic technique was presented for constructing generalized quadrangles. This technique was refined by Payne [3] in order to simplify calculations and search for new quadrangles. All the known generalized quadrangles with parameters q^2, q are described in [2] and [3], but only the aforementioned new family was found using the method in [2]. In this note we will use the formulation in [3] in order to obtain additional quadrangles:

(1.1) **Theorem.** *Let q be a power p^e of an odd prime p. Then*

(i) *If $e > 1$ there are $[(e-1)/2]$ pairwise nonisomorphic generalized quadrangles with parameters q^2, q not isomorphic to any previously known generalized quadrangle; and*

(ii) *If $q > 3$ and $q \equiv \pm 2 \pmod 5$ then there is a generalized quadrangle with parameters q^2, q not isomorphic to any quadrangle in* (i) *nor to any previously known generalized quadrangle.*

Each of the quadrangles in (1.1) admits an automorphism group of order q^5 fixing one point x and transitive on the q^5 points not collinear with x. The quadrangles in (1.1i) have two interesting features. One is their number. The other is the fact that, for every point y collinear with x, there are q automorphisms acting as "elations with center y": automorphisms fixing every point collinear with y.

One other interesting aspect of these quadrangles and of those in [2] is simply their parameters. A generalized quadrangle with parameters s, q necessarily has $s \leqq q^2$, with a great deal of combinatorial information implied by equality (see, e.g., [1]). This tightness makes the number of examples in (1.1 i) seem somewhat unexpected.

The general results contained in [2] and [3] are summarized in § 2. After some preliminary remarks in § 3, we construct the quadrangles in (1.1) in the remaining sections of the paper.

* This research was supported in part by NSF Grant DMS-8320149

2. Construction Procedure

As in [2], let Q be a finite group, and let $\mathscr{F}$ be a family of subgroups of Q. With each $A \in \mathscr{F}$ is associated another subgroup A^*. These are subject to the following conditions: *for each 3-element subset* $\{A, B, C\}$ *of* $\mathscr{F}$, *and some integers s and t,*

(i) $|Q| = s^2 t$, $|\mathscr{F}| = t+1$, $|A| = s$, $|A^*| = st$, $1 < A < A^*$,
(ii) $Q = A^*B$, $A^* \cap B = 1$, and
(iii) $AB \cap C = 1$.

(2.1) *Construction.* Let $A \in \mathscr{F}$ and $q \in Q$ be arbitrary.

Point. Symbol $[\mathscr{F}]$; coset A^*q; element q.

Line. Symbol $[A]$; coset Aq.

Incidence. $[A]$ is on $[\mathscr{F}]$ and A^*q; all other incidences are obtained via inclusion.

By [2], the resulting geometry $\mathscr{Q}(Q, \mathscr{F})$ *is a generalized quadrangle with parameters s, t.*

Payne [3] has used pp. 215–217 of [2] in order to formulate a situation in which (2.1) can be applied. The following is only superficially different from [3, §VI].

Let $F = GF(q)$. For u, $v \in F^2$, uv^t is just the usual dot product. Define a group Q by

(2.2)
$$Q = F^2 \times F \times F^2$$
$$(u, c, v)(u', c', v') = (u + u', c + c' + vu'^t, v + v').$$

Then $|Q| = q^5$ and $Z(Q) = 0 \times F \times 0$.

In order to define $\mathscr{F}$ we assume that, for each $r \in F$, we are given a 2×2 matrix B_r. Write $M_r = B_r + B_r^t$ and

(2.3)
$$A(\infty) = 0 \times 0 \times F^2$$
$$A^*(\infty) = 0 \times F \times F^2$$
$$A(r) = \{(u, uB_r u^t, uM_r) \mid u \in F^2\}$$
$$A^*(r) = \{(u, c, uM_r) \mid u \in F^2\}.$$

Then $A(x)$ is a subgroup of order q^2, and $A^*(x)$ has order q^3, for each $x \in GF(q) \cup \{\infty\}$.

Now assume that B_r *and* M_r *satisfy the following conditions for all distinct r, s,* $z \in F$:

(2.4) (i) $M_r - M_s$ is nonsingular,
(ii) $B_r - B_s$ is anisotropic (i.e., $u(B_r - B_s)u^t = 0 \Rightarrow u = 0$)
and
(iii) $(M_r - M_z)^{-1}(B_r - B_z)(M_r - M_z)^{-1} + (M_z - M_s)^{-1}(B_z - B_s)(M_z - M_s)^{-1}$
is anisotropic.

In general, $D=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is anisotropic $\Leftrightarrow ax^2+(b+c)x+d$ is irreducible over F. When q is odd, D is anisotropic $\Leftrightarrow D+D^t$ is anisotropic $\Leftrightarrow$ $-\det(D+D^t)$ is a nonsquare. Since the matrix in (2.4 iii) plus its transpose is just $(M_r-M_z)^{-1}(M_r-M_s)(M_z-M_s)^{-1}$, it follows that

(2.5) When q is odd, (2.4) is equivalent to the single condition that $-\det(M_r-M_s)$ is a nonsquare for $r \neq s$.

In [3] it is shown that (2.1) *applies to*

(2.6) $$\mathscr{F}=\{A(x) \mid x \in F \cup \{\infty\}\}$$

if and only if (2.4 i–iii) *hold.* We will describe two ways to obtain suitable matrices B_r and M_r in (2.4) or (2.5). These will produce new families of generalized quadrangles with parameters q^2, q.

3. Preliminary Properties

In this section we will describe some simple properties of the groups Q and $A(r)$, and the generalized quadrangle $\mathscr{Q}=\mathscr{Q}(Q, \mathscr{F})$ determined by (2.1)–(2.6).

Lemma 3.1. (i) $[\mathscr{F}]^{\mathrm{Aut}\,\mathscr{Q}}$ *is either* $[\mathscr{F}]$, *all points of a line through* $[\mathscr{F}]$, *or all points of* $\mathscr{Q}$.

(ii) $Q \leqq (\mathrm{Aut}\,\mathscr{Q})_{[\mathscr{F}]}$.

(iii) *If* $A \in \mathscr{F}$ *then* A^* *is the pointwise stabilizer of* $[A]$ *in* Q.

(iv) Aut Q *acts* $GF(q)$*-semilinearly on* $Q/Z(Q) \cong F^4$.

Proof. (i) Q is already transitive on the points not collinear with $[\mathscr{F}]$. Assume that $G=\mathrm{Aut}\,\mathscr{Q}$ moves $[\mathscr{F}]$ but is not point-transitive. Then $[\mathscr{F}]$ can only move to points p collinear with $[\mathscr{F}]$. Since $[\mathscr{F}]^G$ consists of points collinear with both $[\mathscr{F}]$ and p, while Q is transitive on the q^2 points $\neq \mathscr{F}$ on $[A(x)]$, this proves (i)

(ii) Let $U_{[\mathscr{F}], A^*}$ consist of all automorphisms of $\mathscr{Q}$ fixing each line on $[\mathscr{F}]$ or A^* and every point on the line $[A]$ through $[\mathscr{F}]$ and A^*. Then $|U_{[\mathscr{F}], A^*}| \leqq q^2$. Also, $A \leqq U_{[\mathscr{F}], A^*}$ (since $A^* \lhd Q$ and A^* is abelian), and $U_{[\mathscr{F}], A^{*g}}$ is conjugate to $U_{[\mathscr{F}], A^*}$ in Q for each $g \in Q$.

(iii) Clear.

(iv) See [2, p. 217]. □

Lemma 3.2. *Assume that* $x \mapsto B_x$ *is an additive map from* F *to* 2×2 *matrices.*

(a) *Conditions* (2.4 i–iii) *and* (2.5) *become*

(i) M_r *is nonsingular for* $r \neq 0$,

(ii) B_r *is anisotropic for* $r \neq 0$,

(iii) $M_r^{-1} B_r M_r^{-1} + M_s^{-1} B_s M_s^{-1}$ *is anisotropic whenever* r, s, $r+s \neq 0$,

(2.5′) (*for* q odd) $-\det M_r$ *is a nonsquare for* $r \neq 0$.

(b) *The mappings* $(u,c,v)\mapsto(u,c+uB_ru^t,\ v+uM_r)$ *for* $r\in F$ *form a group of* q *automorphisms of* Q *that fixes* $\mathscr{F}$, *is transitive on* $\mathscr{F}-\{A(\infty)\}$, *and induces a group of automorphisms of* $\mathscr{Q}$ *fixing every point collinear with* $A^*(\infty)$.

(c) $(\mathrm{Aut}\,\mathscr{Q})_{[\mathscr{F}]}$ *has at most* 4 *orbits on points.*

Proof. (a) Clear.

(b) A calculation shows that the mapping is an automorphism of Q sending $A(s)$ to $A(s+r)$, and hence inducing an automorphism of $\mathscr{Q}$. Also, each element of $A^*(\infty)$ is fixed, as is each element of $Q/A^*(\infty)$. Since the points collinear with $A^*(\infty)$ have the form $[\mathscr{F}]$, $A^*(\infty)g$ or h with $g\in Q$ and $h\in A^*(\infty)$, this proves (b).

(c) Since Q is transitive on the points $\neq[\mathscr{F}]$ on each line through $[\mathscr{F}]$, (c) follows from (b). □

4. Field Automorphisms

We can now prove the following

(4.1) **Theorem.** *Let* q *be an odd prime power, let* m *be a nonsquare in* $F=GF(q)$, *and let* $\sigma\in\mathrm{Aut}\,F$.

(i) *The matrices* $B_r=\begin{pmatrix} r & 0\\ 0 & -mr^\sigma\end{pmatrix}$, $r\in F$, *determine a generalized quadrangle* $\Pi(\sigma)$ *via* (2.1)-(2.6).

(ii) $\Pi(1)$ *is the* $PSU(4,q)$ *quadrangle.*

(iii) $\Pi(\sigma)\cong\Pi(\tau)\Leftrightarrow\tau=\sigma^{\pm1}$.

Proof. (i) We will use (3.2a): trivially, $M_r=\begin{pmatrix} 2r & 0\\ 0 & -2mr^\sigma\end{pmatrix}$ is anisotropic since $-\det M_r=4mrr^\sigma$ is a nonsquare.

(ii) Here $B_r=r\begin{pmatrix} 1 & 0\\ 0 & -m\end{pmatrix}$. Set $x_1=0$, $x_0=m$ in [3, p. 731].

(iii) If $s\in F^*$ and $S=\begin{pmatrix} 1 & 0\\ 0 & s\end{pmatrix}$ then $(u,\ c,\ v)\to(uS^{-1},\ c,\ vS)$ is an automorphism of Q, and sends $A(r)$ to the group corresponding to the new choice $B_r=\begin{pmatrix} r & 0\\ 0 & -ms^2r^\sigma\end{pmatrix}$. Thus, $\Pi(\sigma)$ does not depend on the choice of m. Replace m by $m^{-\sigma}$ and apply σ^{-1} throughout the definition of $A(r)$ in order to see that $\Pi(\sigma)\cong\Pi(\sigma^{-1})$.

Now suppose that φ: $\Pi(\sigma)\to\Pi(\tau)$ is an isomorphism. By (3.1i) we may assume that (with an obvious notation) $[\mathscr{F}_\sigma]^\varphi=[\mathscr{F}_\tau]$. By (3.1ii), φ conjugates Q to itself. By (3.2b) we may assume that φ sends $A_\sigma^*(\infty)$ to $A_\tau^*(\infty)$ and $A_\sigma^*(r)$ to $A_\tau^*(r')$ for a permutation $r\mapsto r'$ of F. By (3.1iv), φ induces a semilinear transformation

$$(u,v)\mapsto(u,v)^\theta\begin{pmatrix} C & 0\\ 0 & D\end{pmatrix}$$

of $Q/Z(Q)\cong F^2\oplus F^2$ (since we can view $\bar A_\sigma(\infty)=\bar A_\tau(\infty)$ as $0\oplus F^2$ and $\bar A_\sigma(0)=\bar A_\tau(0)$ as $F^2\oplus 0$).

Write $M_r = \begin{pmatrix} 2r & 0 \\ 0 & -2mr^{\sigma} \end{pmatrix}$ as before, and let $N_r = \begin{pmatrix} 2r & 0 \\ 0 & -2mr^{\tau} \end{pmatrix}$. Then there is a mapping $u \mapsto u'$ of $F \to F$, depending on r, such that

$$(u, uM_r)^{\theta} \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} = (u', u'N_{r'})$$

for all u. Clearly, $u' = u^{\theta} C$ and $u'N_{r'} = u^{\theta} M_r^{\theta} D$, so that $u^{\theta} C N_{r'} = u^{\theta} M_r^{\theta} D$. Thus $C N_{r'} = M_r^{\theta} D$. Now $D N_{1'}^{-1} N_{r'} D^{-1} = (M_1^{-1} M_r)^{\theta}$, and a simple calculation shows that either $\tau = \sigma$ or $\tau = \sigma^{-1}$. □

Remark 1. Let H be the group of order q in (3.2b). Then QH is a group of order q^6. Let R be the stabilizer in QH of the point $A^*(\infty)$. Then $R = A^*(\infty) \oplus H$ is elementary abelian of order q^4, and acts regularly on the q^4 lines disjoint from $[A^*(\infty)]$ and fixes all points on $[A(\infty)]$. Consequently, $\mathcal{Q}(Q, \mathcal{F}) \cong \mathcal{Q}(R, \tilde{\mathcal{F}})$ where $\tilde{\mathcal{F}}$ consists of the q^2+1 groups

$$\tilde{A}(\infty) = 0 \times F \times 0 = Z(Q),$$

$$\tilde{A}(u) = \{(0, uB_r u^t, uM_r)\, h_{-1} \mid r \in F\} \qquad \text{for } u \in F^2,$$

while

$$\tilde{A}^*(\infty) = 0 \times F \times F^2 = A^*(\infty),$$

$$\tilde{A}^*(u) = \{(0, vu^t - uB_r u^t, v)\, h_{-r} \mid r \in F,\ v \in F^2\}.$$

This set $\tilde{\mathcal{F}}$ behaves very much like an ovoid in F^4: any three members generate a group of order q^3.

However, if $\sigma \neq 1$ then $\tilde{\mathcal{F}}$ does not determine an inversive plane: $\langle \tilde{A}(0,0), \tilde{A}(1,0), \tilde{A}(1,1) \rangle$ contains just four members of $\tilde{\mathcal{F}}$.

Remark 2. The quadrangles $\Pi(\sigma)$ share one of the properties of the quadrangles in [2]: there is a group of automorphisms fixing the point $[\mathcal{F}]$ and 2-transitive on the lines through $[\mathcal{F}]$. Namely, the automorphisms in (3.2b) induce $A(s) \mapsto A(s+r)$ on $\mathcal{F}$, while the automorphism $(u, c, v) \mapsto (vM_1^{-1}, c - uv^t, -uM_1)$ of Q induces $A(s) \mapsto A(-1/s)$ on $\mathcal{F}$. These automorphisms of $\Pi(\sigma)$ generate a group S inducing $PSL(2,q)$ on $\mathcal{F}$.

Moreover, S induces $SL(2,q)$ on Q since it fixes both $Q_1 = (F \times 0) \times F \times (F \times 0)$ and $Q_2 = (0 \times F) \times F \times (0 \times F)$. (In fact, $S \cong SL(2,q)$ since the generators of S centralize the automorphism $(u, c, v) \mapsto (-u, c, -v)$ of Q.)

Clearly, QS has just three point-orbits on $\Pi(\sigma)$. It follows easily that, if $\sigma \neq 1$, then $\operatorname{Aut} \Pi(\sigma)$ fixes $[\mathcal{F}]$ and $\operatorname{Aut} \Pi(\sigma) \trianglerighteq QS$.

Incidentally, if $i = 1$ or 2 then $\mathcal{F}_i = \{A \cap Q_i \mid A \in \mathcal{F}\}$ determines an $Sp(4,q)$ subquadrangle (on which $Q_i S$ acts in the usual manner). It would be interesting to know whether there are any nonclassical (q,q)-subquadrangles.

Remark 3. All of the above quadrangles (and those in § 5) have q odd. When q is even we have not been able to find any new mappings $r \to B_r$ required in (2.4), except for the following amusing ones: $B_r = \begin{pmatrix} kr & r^{\alpha} \\ r + r^{\alpha} & kr \end{pmatrix}$ where α is any additive isomorphism $F \to F$ and $kx^2 + x + k$ is irreducible. Unfortunately, $uB_r u^t$ and uM_r do not depend on α!

5. An Additional Family

The examples in [2] have $B_r = \begin{pmatrix} -3r & 3r^2 \\ 0 & -r^3 \end{pmatrix}$ (cf. [3, p. 732]). The following is a somewhat similar situation.

(5.1) **Theorem.** *Let q be an odd prime power such that $q \equiv \pm 2$ (mod 5).*

(i) The matrices $B_r = \begin{pmatrix} r/2 & 5r^3 \\ 0 & 10r^5 \end{pmatrix}$, $r \in GF(q)$, determine a generalized quadrangle $\mathcal{Q}$ via (2.1)-(2.6).

(ii) If $q \geqq 7$ then $\mathcal{Q}$ is not isomorphic to any other known quadrangle with parameters q^2, q.

Proof. (i) Here $M_r = \begin{pmatrix} r & 5r^3 \\ 5r^3 & 20r^5 \end{pmatrix}$ and

$$\begin{aligned}\det(M_r - M_s) &= \det\begin{pmatrix} r-s & 5r^3-5s^3 \\ 5r^3-5s^3 & 20r^5-20s^5 \end{pmatrix} \\ &= -5(r^3+2r^2s-2rs^2-s^3)2 \\ &= -5(r-s)^2(r^2+3rs+s^2).\end{aligned}$$

By hypothesis, 5 is a nonsquare in $F = GF(q)$, so that $r^2+3rs+s^2 \neq 0$ for $rs \neq 0$. Thus, $-\det(M_r - M_s)$ is a nonsquare, and hence (2.5) holds.

(ii) If $q=3$ *then* $M_r = \begin{pmatrix} -r & -r \\ 0 & r \end{pmatrix}$. Let $q \geqq 7$. By [3, VI. 5], $\mathcal{Q}$ is not isomorphic to any of the quadrangles in (4.1), nor to any others except, perhaps, for one of the quadrangles in [2].

Our $\bar{\mathcal{F}} = \{A^*(x)/Z(Q) = \bar{A}(x) \mid x \in F \cup \{\infty\}\}$ consists of the 2-spaces $\bar{A}(\infty)$ and

$$\bar{A}(r) = \{(u, uM_r) \mid u \in F^2\}.$$

Thus, $\bar{\mathcal{F}}$ can be viewed as an algebraic variety defined by polynomials of degree 1, 3, and 5. The corresponding variety in [2] is defined by polynomials of degree 1, 2 and 3. Since $q \geqq 7$ these cannot be projectively equivalent - and in fact, not even semilinearly equivalent. In view of (3.1 i, ii, iv), $\mathcal{Q}$ cannot be isomorphic to a quadrangle in [2]. □

In fact, $(\operatorname{Aut} Q)_{\mathcal{F}}$ does not induce $PSL(2, q)$ on $\mathcal{F}$ in (5.1) whereas it does in the case of $\mathcal{F}$ in [2].

References

1. Cameron, P.J.: Partial quadrangles. Q. J. Math., Oxf. **26**, 61–73 (1975)
2. Kantor, W.M.: Generalized quadrangles associated with $G_2(q)$. J. Comb. Theory **A29** 212–219 (1980)
3. Payne, S.E.: Generalized quadrangles as group coset geometries. Congr. Numerantium **19**, 717–734 (1980)

Received November 26, 1984