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Step-by-Step Conjugation of p-Subgroups of a Group

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Alperin [1] has recently introduced a fundamental method for conjugating from one p-Sylow subgroup Q of a finite group G to a second p-Sylow subgroup P in a series of steps. The importance of this is that it provides information concerning the conjugacy in G of subsets of P, that is, fusion of subsets of P. In certain situations it is possible to extend these results to the case of conjugate p-subgroups Q and P of G which are not p-Sylow subgroups of G.

The situation of prime importance is that of a p-Sylow subgroup P of a subgroup of G. That is, P is a p-Sylow subgroup of the stabilizer of a point in transitive permutation representation of G. For example, our results apply to the case of a p-Sylow subgroup of the stabilizer of a point in a 2-transitive permutation group. In general, we are able to handle groups having a sufficiently tight structure, such as groups having a split (B, N)-pair [4]. Moreover, solvable groups can be constructed in which the type of step-by-step conjugation we consider need not apply to p-subgroups other than p-Sylow subgroups.

When P is a p-Sylow subgroup of G, Alperin [1] has indicated many applications of his results to transfer. As we are concerned with p-subgroups that are not necessarily p-Sylow subgroups, our results seem most applicable to fusion. In fact, the significance of our results on conjugation is that they lead in a natural way to new results on fusion. We were thus led in §3 to possibly unexpected results on fusion; most of these can, however, be easily proved by direct methods.

Our notation is all standard. $A \subset B$ denotes that A is a subset of B, while $A \leqslant B$ denotes that A is a subgroup of the group B. $A < B$ will mean $A \leqslant B$ but $A \neq B$. If A is a subset of a group G, $N_G(A)$ is the normalizer of A in G and $C_G(A)$ is the centralizer of A in G.

All groups considered will be finite.

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1. THE RELATION $\sim$

Let G be a group and P a fixed p-subgroup of G.

DEFINITION 1.1. Let Q be conjugate to P in G. The intersection $Q \cap P$ is said to be tame provided that $P \cap N_G(Q \cap P)$ and $Q \cap N_G(Q \cap P)$ are each maximal intersections of conjugates of P with $N_G(Q \cap P)$. That is, if $P^y \cap N_G(Q \cap P) \geqslant P \cap N_G(Q \cap P)$ and $P^z \cap N_G(Q \cap P) \geqslant Q \cap N_G(Q \cap P)$, then $P^y \cap N_G(Q \cap P) = P \cap N_G(Q \cap P)$ and $P^z \cap N_G(Q \cap P) = Q \cap N_G(Q \cap P)$.

When $Q \cap P = 1$ or $Q = P$, the conditions of Definition 1.1 are automatically satisfied. If $Q \cap P$ is maximal among intersections of distinct conjugates of P, then $Q \cap P$ is easily seen to be tame. It should be noted that the definition is dependent of the particular groups Q and P, not just the group $Q \cap P$.

LEMMA 1.2. *If $Q \cap P$ is a tame intersection and H, K are subgroups of G with P p-Sylow in H and Q p-Sylow in K, then $N_P(Q \cap P)$ is p-Sylow in $N_H(Q \cap P)$ and $N_Q(Q \cap P)$ is p-Sylow in $N_K(Q \cap P)$.*

Proof. By symmetry we need only show that $N_P(Q \cap P)$ is p-Sylow in $N_H(Q \cap P)$. If this is not the case, let $R_1 > N_P(Q \cap P)$ be p-Sylow in $N_H(Q \cap P)$ and let R be p-Sylow in H with $R \geqslant R_1$. Then $R \cap N_G(Q \cap P) > R_1 > P \cap N_G(Q \cap P)$, which contradicts the fact that $Q \cap P$ is tame.

If P is a p-Sylow subgroup of G, then $Q \cap P$ is tame according to Definition 1.1 if and only if $N_P(Q \cap P)$ and $N_Q(Q \cap P)$ are each p-Sylow in $N_G(Q \cap P)$. This agrees with Alperin [1].

DEFINITION 1.3. Let Q, R be conjugates of P. We write $Q \sim R$ provided that there exist conjugates $Q_1 ,..., Q_n$, $U_1 ,..., U_n$ of P and elements $x_1 ,..., x_n$ of G such that

(a) $Q_i \cap P$ is tame intersection, $i = 1,..., n$.

(b) $Q_i \cap P \leqslant U_i$, $i = 1,..., n$.

(c) $x_i \in U_i \cap N_G(Q_i \cap P)$, $i = 1,..., n$.

(d) $Q^x = R$, where $x = x_1 ,..., x_n$.

(e) $Q \cap P \leqslant Q_1 \cap P$ and

$$(Q \cap P)^{x_1 \cdots x_i} \leqslant Q_{i+1} \cap P, \qquad i = 1,..., n-1.$$

If $Q \sim R$ and x is as in Definition 1.2, we say that $Q \sim R$ via x. Condition

(e) says that we conjugate from Q to R in a sequence of steps, each of which keeps track of the intersection $Q \cap P$. Moreover, by condition (c) each conjugation is performed by an element in some conjugate of P.

The relation $\sim$ is reflexive but not necessarily symmetric. If $Q \sim R$ is false we write $Q \not\sim R$.

For completeness we state and prove three lemmas of Alperin [1].

LEMMA 1.4. *If Q, R, S are conjugates of P such that $Q \sim R$ and $R \sim S$, then $Q \sim S$.*

Proof. Suppose Q_i, U_i, x_i, $i = 1,..., m$ and R_i, V_i, y_i, $i = 1,..., n$, yield $Q \sim R$ and $R \sim S$ as in Definition 1.1. Set $l = m + n$ and

$$S_i = \begin{cases} Q_i \\ R_{i-m} \end{cases}, \qquad W_i = \begin{cases} U_i \\ V_{i-m} \end{cases}, \qquad \text{and} \qquad z_i = \begin{cases} x_i & i = 1,..., m \\ y_{i-m} & i = m+1,..., l. \end{cases}$$

We claim that S_i, W_i, z_i, $i = 1,..., l$ yield $Q \sim S$.

Clearly $S_i \cap P$ is a tame intersection for $i = 1,..., l$; thus (a) holds. Also (b) and (c) hold. If $z = z_1 \cdots z_l$, then $Q^z = (Q^{x_1 \cdots x_m})^{y_1 \cdots y_n} = R^{y_1 \cdots y_n} = S$, so that (d) holds.

By definition, $Q \cap P \leqslant Q_1 \cap P = S_1 \cap P$. If $i = 1,..., m - 1$, then

$$(Q \cap P)^{z_1 \cdots z_i} = (Q \cap P)^{x_1 \cdots x_i} \leqslant Q_{i+1} \cap P = S_{i+1} \cap P.$$

Also

$$\begin{aligned}(Q \cap P)^{z_1 \cdots z_m} &= ((Q \cap P)^{x_1 \cdots x_{m-1}})^{x_m} \leqslant (Q_m \cap P)^{x_m} \cap R \\ &= Q_m \cap P \cap R \leqslant R \cap P \leqslant R_1 \cap P = S_{m+1} \cap P.\end{aligned}$$

Finally, for $i = m + 1,..., l - 1$,

$$\begin{aligned}(Q \cap P)^{z_1 \cdots z_i} &= ((Q \cap P)^{x_1 \cdots x_m})^{y_1 \cdots y_{i-m}} \leqslant (R \cap P)^{y_1 \cdots y_{i-m}} \\ &\leqslant R_{i-m+1} \cap P = S_{i+1}.\end{aligned}$$

This proves the claim.

LEMMA 1.5. *Let Q, R be conjugates of P such that $R \cap P \geqslant Q \cap P$, $R \sim P$ via x, and $Q^x \sim P$. Then $Q \sim P$.*

Proof. By Lemma 1.4 it suffices to show that $Q \sim Q^x$. Let Q_i, U_i, x_i, $i = 1,..., n$, yield $R \sim P$, with $x = x_1 \cdots x_n$. We claim that $Q \sim Q^x$ via x. For, $Q \cap P \leqslant R \cap P$ implies that $Q \cap P \leqslant R \cap P \leqslant Q_1 \cap P$ and

$$(Q \cap P)^{x_1 \cdots x_i} \leqslant (R \cap P)^{x_1 \cdots x_i} \leqslant Q_{i+1} \cap P,$$

for $i = 1,..., n - 1$.

LEMMA 1.6. *Let Q, R be conjugates of P such that $Q \cap R > Q \cap P$ and $R \sim P$. Suppose further that $S \sim P$ for each conjugate S of P with $|S \cap P| > |Q \cap P|$. Then $Q \sim P$.*

Proof. Let $R \sim P$ via x, so that $R^x = P$. Then $Q^x \cap P = Q^x \cap R^x = (Q \cap R)^x$. By hypothesis we then have $|Q^x \cap P| = |Q \cap R| > |Q \cap P|$, so that $Q^x \sim P$. As $R \cap P \geqslant (Q \cap R) \cap P \geqslant Q \cap P$ and $R \sim P$, the result follows from Lemma 1.5.

LEMMA 1.7. *Let Q be a conjugate of P such that $Q \cap P$ is a tame intersection, $Q \not\sim P$, but $S \sim P$ wherever S is a conjugate of P with $|S \cap P| > |Q \cap P|$. Let $x \in \langle P_0, Q_0 \rangle$,* where *$P_0, Q_0$ are subgroups of $N_G(Q \cap P)$ containing $Q \cap P$ and conjugate in G to subgroups of P.*

(i) *If R is a conjugate of P such that $R \cap P = Q \cap P$, then $R \sim R^x$.*

(ii) *$Q^x \not\sim P$, $Q^x \cap P = Q \cap P$, and $Q^x \cap P$ is a tame intersection.*

Proof. Let $x = x_1 \cdots x_n$, $x_i \in P_0$ or $x_i \in Q_0$ for $i = 1,..., n$. Let $P_0 \leqslant V$, $Q_0 \leqslant W$ where V, W are conjugates of P. Set $Q_i = Q$, $i = 1,..., n$. Set $U_i = V$ if $x_i \in P_0$ and $U_i = W$ if $x_i \notin P_0$. Then $x_i \in U_i \cap N_G(Q \cap P)$, $i = 1,..., n$, $R \cap P = Q_1 \cap P$ and $(R \cap P)^{x_1 \cdots x_i} = Q_{i+1} \cap P$, $i = 1,..., n-1$. This yields $R \sim R^x$, via x, proving (i).

Now set $R = Q$. Since $Q \sim Q^x$ and $Q \not\sim P$, we have $Q^x \not\sim P$. Since $x \in N_G(Q \cap P)$, $Q^x \cap P \geqslant (Q \cap P)^x \cap P = Q \cap P$. Thus $|Q^x \cap P| \geqslant |Q \cap P|$. As $Q^x \not\sim P$, by hypothesis we must have $|Q^x \cap P| = |Q \cap P|$, so that $Q^x \cap P = Q \cap P$.

It remains to show that $Q^x \cap P$ is a tame intersection. Since $N_G(Q^x \cap P) = N_G(Q \cap P)$ and since $Q \cap P$ is a tame intersection, $P \cap N_G(Q^x \cap P)$ is a maximal intersection of a conjugate of P with $N_G(Q^x \cap P)$. Suppose that $S \cap N_G(Q^x \cap P) > Q^x \cap N_G(Q^x \cap P)$, where S is a conjugate of P. Then

$$S \cap N_G(Q \cap P) > Q^x \cap N_G(Q \cap P) = Q^x \cap N_G(Q \cap P)^x,$$

so that

$$S^{x^{-1}} \cap N_G(Q \cap P) > Q \cap N_G(Q \cap P).$$

This contradicts the fact that $Q \cap P$ is a tame intersection, proving (ii).

THEOREM 1.8. *Let P be a subgroup of a finite group G. Suppose that there is a conjugate T of P for which $T \not\sim P$. Then there is a conjugate Q of P such that*

(i) *$Q \not\sim P$, whereas $S \sim P$ whenever S is a conjugate of P with $|S \cap P| > |Q \cap P|$.*

(ii) *$Q \cap P$ is a tame intersection and $Q \neq P$.*

(iii) *$\langle N_P(Q \cap P), N_Q(Q \cap P)\rangle$ is a p-group.*

(iv) *Let R be a conjugate of P containing $Q \cap P$ and H, K, L subgroups of G such that P, Q, R are p-Sylow in H, K, L, respectively. If $Q \cap P$ is not p-Sylow in $K \cap L$, then $R \not\sim P$.*

(v) *Let R be a conjugate of P containing $Q \cap P$ and H, K, L subgroups of G such that P, Q, R are p-Sylow in H, K, L, respectively. If $Q \cap P$ is not p-Sylow in $K \cap L$, then $R \cap P = Q \cap P$ and this intersection is p-Sylow in $L \cap H$.*

(vi) *Let P be p-Sylow in a subgroup H of G and Q be p-Sylow in a subgroup K of G. Then $Q \cap P$ is p-Sylow in $K \cap H$.*

Proof. Let T be a conjugate of P such that $T \not\sim P$ and $|T \cap P|$ is maximal for such groups T. Then $T \neq P$, and $S \sim P$ for every conjugate of P such that $|S \cap P| > |T \cap P|$.

Let S be a conjugate of P containing $P \cap N_G(T \cap P)$ such that $S \cap N_G(T \cap P)$ is a maximal intersection of $N_G(T \cap P)$ with a conjugate of P. Then $S \cap P > P \cap N_G(T \cap P) > T \cap P$ implies that $S \sim P$. Let $S \sim P$ via x. By Lemma 1.5, $T^x \not\sim P$. Moreover, $P = S^x \geqslant (T \cap P)^x$, so that $T^x \cap P \geqslant (T \cap P)^x \cap P = (T \cap P)^x$. If $T^x \cap P > (T \cap P)^x$, then $T^x \sim P$ as $|T^x \cap P| > |T \cap P|$, a contradiction. Thus, $T^x \cap P = (T \cap P)^x$. We claim that $P \cap N_G(T^x \cap P)$ is a maximal intersection of $N_G(T^x \cap P)$ with a conjugate of P. For, if V is a conjugate of P such that $V \cap N_G(T^x \cap P) > P \cap N_G(T^x \cap P)$, then

$$V \cap N_G(T \cap P)^x = V \cap N_G(T^x \cap P) > P \cap N_G(T \cap P)^x,$$

so that $V^{x^{-1}} \cap N_G(T \cap P) > P^{x^{-1}} \cap N_G(T \cap P) = S \cap N_G(T \cap P)$, contradicting the choice of S.

Let U be a conjugate of P containg $T^x \cap N_G(T^x \cap P)$ such that $U \cap N_G(T^x \cap P)$ is a maximal intersection of $N_G(T^x \cap P)$ with a conjugate of P. We first note that $U \not\sim P$. For otherwise,

$$T^x \cap U \geqslant T^x \cap N_G(T^x \cap P) > T^x \cap P$$

and Lemma 1.6 implies that $T^x \sim P$, a contradiction. Clearly, $U \cap P \geqslant T^x \cap P$ and $|U \cap P| \geqslant |T^x \cap P| = |(T \cap P)^x| = |T \cap P|$. Then $U \cap P = T^x \cap P$, as otherwise the maximality of $|T \cap P|$ would imply that $U \sim P$. As $U \cap N_G(T^x \cap P) = U \cap N_G(U \cap P)$ is, by definition, a maximal intersection of $N_G(T^x \cap P) = N_G(U \cap P)$ with a conjugate of P, and $P \cap N_G(T^x \cap P) = P \cap N_G(U \cap P)$ is known to be a maximal intersection of $N_G(U \cap P)$ with a conjugate of P, $U \cap P$ is a tame intersection.

Let $\bar{P}$ be a p-Sylow subgroup of $\langle N_P(U \cap P), N_U(U \cap P)\rangle$ containing $N_P(U \cap P)$. Let $N_U(U \cap P)^v \leqslant \bar{P}$, $v \in \langle N_P(U \cap P), N_U(U \cap P)\rangle$. If we set $Q = U^v$, then Lemma 1.7 implies that $U \sim Q$, $Q \not\sim P$, $U \cap P = Q \cap P$ and $Q \cap P$ is a tame intersection.

We now verify (i)–(vi).

(i) We have already observed that $Q \not\sim P$. Moreover, $|Q \cap P| = |U \cap P| = |T \cap P|$, so that $S \sim P$ whenever S is a conjugate of P for which $|S \cap P| > |Q \cap P|$.

(ii) $Q \cap P$ is known to be a tame intersection.

(iii) This follows from

$$N_Q(Q \cap P) = N_U(U \cap P)^v \leqslant \bar{P} \quad \text{and} \quad N_P(Q \cap P) = N_P(U \cap P) \leqslant \bar{P}.$$

(iv) Let R_1 be a p-Sylow subgroup of L such that $N_{R_1}(Q \cap P)$ is p-Sylow in $N_L(Q \cap P)$ and $N_{R_1}(Q \cap P) \geqslant N_R(Q \cap P)$. Let Q_0 be a p-Sylow subgroup of $K \cap L \cap N_G(Q \cap P)$, so that $Q_0 > Q \cap P$. By Lemma 1.2, $Q_0 \leqslant N_Q(Q \cap P)^y$ for some $y \in \langle Q_0, N_Q(Q \cap P)\rangle$. By Lemma 1.7, $Q \sim Q^y \not\sim P$, $Q^y \cap P = Q \cap P$ and $Q^y \cap P$ is a tame intersection. Also $Q_0{}^z \leqslant N_{R_1}(Q \cap P)$ for some $z \in \langle Q_0, N_{R_1}(Q \cap P)\rangle$. By Lemma 1.7, $Q^y \sim Q^{yz} \not\sim P$ and $Q^{yz} \cap P = Q^y \cap P = Q \cap P$. Since

$$Q^{yz} \cap R_1 \geqslant Q_0{}^z \cap R_1 = Q_0{}^z > (Q \cap P)^z = Q \cap P = Q^{yz} \cap P$$

and $Q^{yz} \not\sim P$, by Lemma 1.6 $R_1 \not\sim P$. Then $R_1 \cap P \geqslant Q \cap P$ implies that $R_1 \cap P = Q \cap P$. Finally, $R_1 \cap R \geqslant N_R(Q \cap P) > Q \cap P = R_1 \cap P$, so that another application of Lemma 1.6 yields $R \not\sim P$.

(v) Assume that $Q \cap P$ is not p-Sylow in $K \cap L$. Let R_1 be as in the proof of (iv). By(iv), $R \not\sim P$. As $R \cap P \geqslant Q \cap P$ it follows that $R \cap P = Q \cap P$.

Suppose that $Q \cap P$ is not p-Sylow in $L \cap H$. Let P_0 be a p-Sylow subgroup of $L \cap H \cap N_G(Q \cap P)$. Then $P_0 > Q \cap P$. As $N_{R_1}(Q \cap P)$ is p-Sylow in $N_L(Q \cap P)$, there is an element $t \in \langle P_0, N_{R_1}(Q \cap P)\rangle$ such that $P_0 \leqslant N_{R_1}(Q \cap P)^t$. By Lemma 1.2, there is an element $w \in \langle P_0, N_P(Q \cap P)\rangle$ such that $P_0{}^w \leqslant N_P(Q \cap P)$. Then $R_1 \sim R_1{}^t$ by Lemma 1.7, so that $R_1{}^t \not\sim P$. As $R_1{}^t \cap P \geqslant Q \cap P$ we have $R_1{}^t \cap P = Q \cap P$. By another application of Lemma 1.7, $R_1{}^t \sim R_1^{tw}$, so that $R_1^{tw} \not\sim P$. However,

$$|R_1^{tw} \cap P| \geqslant |P_0{}^w \cap P| = |P_0{}^w| > |(Q \cap P)^w| = |Q \cap P|$$

implies that $R_1^{tw} \sim P$, a contradiction.

(vi) Set $R = Q$ and $L = K$ in (v).

This completes the proof of Theorem 1.8.

We note that, in Theorem 1.8 (iv), (v) and (vi), we may very well be considering several permutation representations simultaneously.

As a consequence of the above theorem, we have the following result of Alperin ([1], Lemma 5)

COROLLARY 1.9. (Alperin). *If P is a p-Sylow in G and if Q is any conjugate of P, then $Q \sim P$.*

Proof. In Theorem 1.8 (vi) set $H = K = G$.

The situation considered in the following theorem is one of the simplest to which Theorem 1.8 applies. As the proof illustrates, a fusion theorem can be obtained whenever P is a p-subgroup of G such that $Q \sim P$ for all conjugates Q of P.

THEOREM 1.10. *Let G be a finite group, H a subgroup of G, and P a p-Sylow subgroup of H. Suppose that each intersection of the form $H^g \cap H$ contains a p-Sylow subgroup of H. Let A and A^z be subsets of P conjugate in G. Then there exist conjugates $Q_1, \ldots, Q_n$, $U_1, \ldots, U_n$ of P and elements $x_1, \ldots, x_m, y$ of G such that*

(a) $z = x_1 \cdots x_n y$.

(b) *$Q_i \cap P$ is a tame intersection, $i = 1, \ldots, n$.*

(c) *x_i is in $U_i \cap N_G(Q_i \cap P)$, $i = 1, \ldots, n$, and y is in $N_G(P)$.*

(d) $Q_i \cap P \leqslant U_i$, $i = 1, \ldots, n$.

(e) *$A \subseteq Q_1 \cap P$ and $A^{x_1 \cdots x_i} \subseteq Q_{i+1} \cap P$, $i = 1, \ldots, n-1$.*

We remark that, in Theorem 1.10, we have $A^{x_1 \cdots x_i}$, $A^{x_1 \cdots x_{i+1}}$ contained in $Q_{i+1} \cap P$ and conjugate by the element x_{i+1} of $N_G(Q_{i+1} \cap P)$, $i = 1, \ldots, n-1$.

Proof. We first note that, by Theorem 1.8 (vi), we have $P^{z^{-1}} \sim P$. Let $P^{z^{-1}} \sim P$ via x and choose conjugates $Q_1, \ldots, Q_n$, $U_1, \ldots, U_n$ of P and elements $x_1, \ldots, x_n$ in G in accordance with Definition 1.3. $P^{z^{-1}x} = P$, so setting $y = x^{-1}z$ we have y in $N_G(P)$ and $z = xy = x_1 \cdots x_n y$. This gives (a), (b), (c), (d) of Theorem 1.10. Since $A \subseteq P^{z^{-1}} \cap P \leqslant Q_1 \cap P$ and

$$A^{x_1 \cdots x_i} \subseteq (P^{z^{-1}} \cap P)^{x_1 \cdots x_i} \leqslant Q_{i+1} \cap P, \qquad i = 1, \ldots, n-1$$

(e) also holds.

This completes the proof.

As a particular case, if P is p-Sylow in G, Theorem 1.10, yields Alperin's Main Theorem [1].

2. EXAMPLES

In this section we list some examples of situations in which Theorem 1.8 is applicable. In each case, our conclusion will be that $Q \sim P$ for all conjugates Q of P. Hence, in each case there is a fusion Theorem as in Theorem 1.10 and the remark preceding its proof.

EXAMPLE 1. Let G be a transitive permutation group on a finite set Ω such that each orbit of G_α, $\alpha \in \Omega$, has length relatively prime to p. Let P be a p-Sylow subgroup of $H = G_\alpha$.

EXAMPLE 1′. Let G be a finite group such that $G = BN_G(P)B$, where B is a subgroup of G and P is a p-Sylow subgroup of B.

Example 1 is precisely the situation already considered in Theorem 1.10. Examples 1 and 1′ are easily seen to describe the same situation.

EXAMPLE 2. Let G be a finite group having a normal subgroup M such that G/M has a split (B, N)-pair (see [4] for definitions). Let $G/M = B/M \cdot N/M \cdot B/M$ in accordance with the (B, N) structure of G/M, and let H/M be an abelian Hall subgroup of B/M with $N/M \leqslant N_{G/M}(H/M)$. Suppose that $p \mid | H/M |$, and let P be p-Sylow in H.

Since $H \trianglelefteq N$, $N \leqslant HN_G(P)$ by the Frattini argument. Therefore $G = BNB = BHN_G(P)B = BN_G(P)B$.

We now appeal to Example 1′.

As a special case of Example 2 we have

EXAMPLE 3. G is a finite group having a split (B, N)-pair and P is p-Sylow in the abelian Hall subgroup H of B.

EXAMPLE 4. Let $G = GL(V)$, V an n-dimensional vector space over a finite field of characteristic p. Let H be the pointwise stabilizer of a subspace S of V and P a p-Sylow subgroup of H.

Suppose that $Q = P^g$ is as in Theorem 1.8. Let $\bar{P}$ be a p-Sylow subgroup of G containing $N_P(Q \cap P)$ and $N_Q(Q \cap P)$. $\bar{P}$ fixes a nest (or composition series) of subspaces $V_0 \subset V_1 \subset \cdots \subset V_n = V$, $\dim V_i = i$, $i = 1,..., n$. Also, P fixes a chain of subspaces $S = S_s \subset S_{s+1} \subset \cdots \subset S_n = V$, $\dim S_i = i$, $i = s,..., n$, and these are the only subspaces fixed by P of dimension $\geqslant s$.

Set $t = \dim(S + Sg)$; t may be n. As $P^g \cap P$ is p-Sylow in $H^g \cap H$ by Theorem 1.8 (vi), $P^g \cap P$ fixes a chain of subspaces $S + Sg = T_t \subset \cdots \subset T_n$, $\dim T_i = i$, $i = s,..., n$, and these are the only subspaces fixed by $P^g \cap P$ of dimension $\geqslant t$. As $P^g \cap P \leqslant P$, it follows that $T_i = S_i$, $i = t,..., n$. Similarly, $T_i = S_i g$, $i = t,..., n$. As P and P^g stabilize the chain $T_t \subset \cdots \subset T_n$, and $P^g \cap P$ is the only p-Sylow subgroup of $H^g \cap H$ stabilizing this chain, it follows that P and P^g normalize $P^g \cap P$. Then also $P, P^g \leqslant \bar{P}$, so that these groups stabilize the nest $V_0 \subset \cdots \subset V_n$. It follows that $S_i = V_i = S_i g$, $i = s, \ldots, n$. In particular, $S = Sg$. Then both P and P^g are the pointwise stabilizer of $S = Sg$ in $\bar{P}$. Thus, $P = P^g$, a contradiction.

It is clear that the above argument is equally valid for $SL(V)$, $PGL(V)$ and $PSL(V)$.

EXAMPLE 5. Let G be a 2-transitive permutation group on a finite set Ω, and let P be a p-Sylow subgroup of G_α, $\alpha \in \Omega$.

Set $n = |\Omega|$. If $p \nmid n$ then P is p-Sylow in G and Alperin's result (Corollary 1.9) applies. If $p \mid n$ then $p \nmid (n - 1)$ and Example 1 applies.

We now consider the case of a 2-transitive group in which the stabilizer of a point is one of the groups described in Example 1. Theorem 1.8 (v) is needed here.

EXAMPLE 6. Let G be a 2-transitive permutation group on a finite set Ω such that each orbit of $G_{\alpha\beta}$, $\alpha, \beta \in \Omega$, $\alpha \neq \beta$, has length relatively prime to p. Let P be a p-Sylow subgroup of $G_{\alpha\beta}$.

Let Q be a conjugate of P as in Theorem 1.8. By hypothesis, $P \leqslant G_{\alpha\beta\gamma}$ for some $\gamma \neq \alpha, \beta$, and $Q \leqslant G_{\delta\epsilon}$, $\delta \neq \epsilon$. As $\delta \neq \alpha$ or β, we may assume that $\delta \neq \alpha$. Set $H = G_{\alpha\beta}$, $K = G_{\delta\epsilon}$ and $L = G_{\alpha\delta}$. Then $Q \cap P \leqslant G_{\alpha\delta\epsilon} = K \cap L$ and, by hypothesis, $K \cap L$ contains a p-Sylow subgroup R of L containing $Q \cap P$. By Theorem 1.8 (v) it follows that $R \cap P = Q \cap P$ is p-Sylow in $L \cap H = G_{\alpha\beta\delta}$. As $G_{\alpha\beta\delta}$ contains a p-Sylow subgroup of $G_{\alpha\beta}$, we must have $Q = P$, a contradiction.

As in Example 5, we now obtain

EXAMPLE 7. Let G be a finite 3-transitive permutation group and P a p-Sylow subgroup of the stabilizer of two points.

3. APPLICATIONS TO FUSION.

Let P be a p-subgroup of a group G.

DEFINITION 3.1. A subgroup N of G is said to *control fusion* in P provided that two subsets of P conjugate in G are already conjugate in N.

LEMMA 3.2. *Let $P \leqslant G$ be a non-abelian Hamiltonian 2-group and let Q, U be conjugates of P with $Q \cap P \leqslant U$. If $g \in U$, then there is an element $h \in P$ such that $gh^{-1} \in C_G(Q \cap P)$.*

Proof. Let $Q \cap P = CxD$, where D is elementary abelian and C is either quaternion of order 8 or cyclic of order $\leqslant 4$. Then D is centralized by P and U.

If C is quaternion then $U = (Q \cap P) \cdot C_U(Q \cap P)$, which implies the result. If C is cyclic of order $\leqslant 4$ then g either centralizes or inverts $Q \cap P$, and either action can be obtained by the use of an element of P. There is thus an element h in P with $gh^{-1} \in C_G(Q \cap P)$.

THEOREM 3.3. *Let G be a finite group, H a subgroup of G, and P a p-Sylow subgroup of H. Suppose that P is Hamiltonian and that each intersection of the form $P^g \cap P < P$ is not a p-Sylow subgroup of $H^g \cap H$. Then $N_G(P)$ controls fusion in P.*

Proof. By Theorem 1.8 (vi) we have $Q \sim P$ for each conjugate Q of P; and by the remark preceding Theorem 1.10 it follows that the conclusions of Theorem 1.10 hold if A, A^z are subsets of P conjugate in G. Thus, there are conjugates $Q_1, \ldots, Q_n$, $U_1, \ldots, U_n$ of P and elements $x_1, \ldots, x_n$, y of G satisfying (a)–(e) of Theorem 1.10. If P is abelian then x_i centralizes $Q_i \cap P$, $i = 1, \ldots, n$, so that $A^z = A^{x_1 \cdots x_n y} = A^y$; since $y \in N_G(P)$ the result follows. If P is non-abelian then Lemma 3.2 applies, and for each $i = 1, \ldots, n$ there is an element $h_i \in P$ such that $A^{x_1 \cdots x_i} = A^{x_1 \cdots x_{i-1} h_i}$. Then $A^z = A^{x_1 \cdots x_n y} = A^{h_1 \cdots h_n y}$ and $h_1 \cdots h_n y \in N_G(P)$. This completes the proof of the theorem.

The above theorem generalizes a classical result of Burnside [3], p. 327. We note that the hypotheses of Theorem 3.3 imply that $H^g \cap H$ always contains a p-Sylow subgroup of H (see Lemma 3.8). Using this fact and Burnside's method, it is possible to give a direct proof of Theorem 3.3 (see the proof of Theorem 3.7). However, we note that the proofs of Theorems 1.10 and 3.3 show that $N_G(P)$ controls fusion in P whenever P is a Hamiltonian p-subgroup of G and $Q \sim P$ for each conjugate Q of P.

If P is assumed to be a nilpotent Hall subgroup of a subgroup H of G, then much of §1 holds with little change. This is due to theorems of Wielandt [5]. In particular, Theorem 1.8 can be generalized with the exception that part (iii) must be removed. As a result, Theorem 3.3 holds in the more general setting. We mention one special case (see §2, Example 3).

COROLLARY 3.4. *Let G be a group with a split (B, N)-pair and let H be a q-complement in B, where q is the characteristic of the group. Then $N_G(H)$ controls fusion in H.*

This corollary can also be proved directly using the uniqueness of the Bruhat decomposition, although our proof indicates that the only fact needed is $G = BN_G(H)B$.

As in Theorem 3.3, by examining the step-by-step conjugation we obtain the following

THEOREM 3.5. *Let G be a finite group, H a subgroup of G, and P a p-Sylow subgroup of H. Suppose that each intersection of the form $P^g \cap P < P$ is not a* *p-Sylow subgroup of* $H^g \cap H$. *Then* $N_G(P)$ *controls the fusion of maximal subgroups of* P. *That is, two maximal subgroups of* P *conjugate in* G *are already conjugate in* $N_G(P)$.

More generally, the conclusion of Theorem 3.5 holds whenever it is known that $Q \sim P$ for all conjugates Q of P. An interesting special case of this generalization is provided by the situation described in §2, Example 4.

We remark that, in Theorem 3.5, if we had assumed that P is a p-group having a cyclic maximal subgroup of order p^n, then we would conclude that $N_G(P)$ controls the fusion of elements of order p^n in P.

DEFINITION 3.6. If P is a subgroup of G, let $Z^*(P)$ consist of those elements x of P with the property that each conjugate of x lying in P is actually in $Z(P)$.

We were led to the proof of the following theorem by the step-by-step conjugation approach. However, we shall first give a proof by direct methods.

THEOREM 3.7. *Let* P *be* p*-Sylow in a subgroup* H *of* G *such that each intersection of the form* $P^g \cap P < P$ *is not a* p*-Sylow subgroup of* $H^g \cap H$. *Then* $Z^*(P)$ *is a group and* $N_G(Z^*(P))$ *controls fusion in* P.

LEMMA 3.8. (I. M. Isaacs). *Suppose* P, H, G *satisfy the hypotheses of Theorem* 3.7. *Then each intersection* $H^g \cap H$ *contains a* p*-Sylow subgroup of* H.

Proof. Let $g \in G$ and set $Q = P^g$, $K = H^g$. Let R be p-Sylow in $K \cap H$. There are elements $h \in H$, $k \in K$ such that $R \leqslant P^h$ and $R \leqslant Q^k$. If $P^h = Q^k$, then $P^h \leqslant K \cap H$ as required. Suppose that $P^h \neq Q^k$. By hypothesis, there is a p-Sylow subgroup S of $K^{kh^{-1}} \cap H$ properly containing $Q^{kh^{-1}} \cap P$. Then $R \leqslant Q^k \cap P^h = (Q^{kh^{-1}} \cap P)^h < S^h \leqslant (K^{kh^{-1}} \cap H)^h = K \cap H$. As R is p-Sylow in $K \cap H$, this is a contradiction.

Proof of Theorem 3.7. Let A and $A^{g^{-1}}$ be subsets of P conjugate in G. Then $A \subseteq P^g \cap P$. Let R be a p-Sylow subgroup of $H^g \cap H$ containing $P^g \cap P$. By Lemma 3.8, R is p-Sylow in H and H^g. By definition, $C_G(A)$ contains $Z^*(P)$, $Z^*(R)$ and $Z^*(P^g)$. As $Z^*(P)$ and $Z^*(R)$ are in $C_H(A)$, there is an element $c \in C_H(A)$ such that $\langle Z^*(P)^c, Z^*(R)\rangle$ is contained in a p-Sylow subgroup P_1 of H. Since $Z^*(P_1)$ is weakly closed in P_1 with respect to G, it follows that $Z^*(P)^c = Z^*(P_1) = Z^*(R)$. Similarly, $Z^*(R)^d = Z^*(P^g)$ with $d \in C_{H^g}(A)$. Thus, $Z^*(P)^{cd} = Z^*(P)^g$. Set $n = cdg^{-1}$. Then $A^{g^{-1}} = A^{d^{-1}c^{-1}n} = A^n$. It follows that $N_G(Z^*(P))$ controls fusion in P.

To show that $Z^*(P)$ is a group we need only prove that, if $a, b \in Z^*(P)$, then $ab \in Z^*(P)$. Let $(ab)^g \in P$. Since $N_G(Z^*(P))$ controls fusion in P we may assume that $g \in N_G(Z^*(P))$. Then a^g, b^g are in $Z^*(P) \leqslant Z(P)$, so that $(ab)^g = a^g b^g \in Z(P)$.

COROLLARY 3.9. *Let P be a p-Sylow subgroup of a subgroup H of G such that each intersection of the form $P^g \cap P < P$ is not a p-Sylow subgroup of $H^g \cap H$. If the subgroup $\Omega_1(P)$ of P generated by all elements of order dividing p is a subgroup of $Z(P)$, then $N_G(\Omega_1(P))$ controls fusion in P.*

In order to give an alternative proof of Theorem 3.7, we first introduce a conjugation family, in the sense of Alperin [1]. Let P be a p-subgroup of group G. Let $\mathscr{F}$ be the set of pairs $(Q \cap P, T)$ where $Q \cap P$ is a tame intersection and $T = N_G(Q \cap P)$ if $Z^*(P) \subseteq Q \cap P$, while $T = C_G(Q \cap P)$ if $Z^*(P) \not\subseteq Q \cap P$.

For conjugates Q, R of P we define $Q \approx R$ in case there exist elements $(Q_1 \cap P, T_1),..., (Q_n \cap P, T_n)$ of $\mathscr{F}$, conjugates $U_1 ,..., U_n$ of P, and elements $x_1 ,..., x_n$ in G such that

(a) $Q_i \cap P \leqslant U_i, i = 1,..., n.$

(b) $x_i \in U_i \cap T_i, i = 1,..., n.$

(c) $Q^{x_1 \cdots x_n} = R.$

(d) $Q \cap P \leqslant Q_1 \cap P$ and $(Q \cap P)^{x_1 \cdots x_i} \leqslant Q_{i+1} \cap P, i = 1,..., n - 1.$

THEOREM 3.10. *Let P be p-Sylow in a subgroup H of G such that, for each $g \in G$, $H^g \cap H$ contains a p-Sylow subgroup of H. If Q is any conjugate of P, then $Q \approx P$.*

Theorem 3.10 can be proved by using the same methods as in Theorem 1.8 (compare Alperin [1], Theorem 5.1).

Alternative proof of Theorem 3.7. Theorem 3.7 leads to a fusion theorem as in Theorem 1.10. If A, A^z are subsets of P conjugate in G then there are elements $(Q_1 \cap P, T_1),..., (Q_n \cap P, T_n)$ of $\mathscr{F}$, conjugates $U_1 ,..., U_n$ of P, and elements $x_1 ,..., x_n , y$ of G such that $A^z = A^{x_1 \cdots x_n y}$, $y \in N_G(P)$, and $x_i \in U_i \cap T_i$, for $i = 1,..., n$. Suppose that $Z^*(P) \subseteq Q_i \cap P$ and hence $T_i = N_G(Q_i \cap P)$. Then $Z^*(P)^{x_i} \subseteq Q_i \cap P \leqslant P$, so that $Z^*(P)^{x_i} = Z^*(P)$ and $x_i \in N_G(Z^*(P))$. If $Z^*(P) \not\subseteq Q_i \cap P$, then $x_i \in C_G(Q_i \cap P)$ and $A^{x_1 \cdots x_i} = A^{x_1 \cdots x_{i-1}}$. Since $y \in N_G(P) \leqslant N_G(Z^*(P))$, we have $A^z = A^h$ for some $h \in N_G(Z^*(P))$. This proves that $N_G(Z^*(P))$ controls fusion in P. The remaining assertion of Theorem 3.7 is proved as before.

4. CONCLUDING REMARKS

Results such as those proved in Alperin and Gorenstein [2] seem to rely strongly on having a p-Sylow subgroup P of G. However, it seems likely that further fusion results along the lines of Theorems 1.10, 3.3, 3.5, and 3.7 can be obtained. It would also be of interest to have more examples of situations in which $Q \sim P$ for all conjugates Q of P. In the situation of Theorem 1.8, additional information can be obtained concerning the conjugates of P containing $Q \cap P$. For example, we mention without proof the following generalization of Theorem 1.8 (iv) and (v):

Let $Q = R_1, \ldots, R_n$ be conjugates of P containing $Q \cap P$ and $L_1, \ldots, L_n$, H subgroups of G such that R_i is p-Sylow in L_i, $i = 1, \ldots, n$, and P is p-Sylow in H. Suppose that $Q \cap P$ is not p-Sylow in $L_i \cap L_{i+1}$, $i = 1, \ldots, n-1$. Then $R_i \not\sim P$, $R_i \cap P = Q \cap P$, and $Q \cap P$ is p-Sylow in $L_i \cap H$, $i = 1, \ldots, n$.

References

1. J. L. Alperin, "Sylow Intersections and Fusion," *J. Algebra* **6** (1967), 222–241.
2. J. L. Alperin and D. Gorenstein, "Transfer and Fusion in Finite Groups," *J. Algebra* **6** (1967), 242–255.
3. W. Burnside, "Theory of Groups of Finite Order," Dover, New York, 1955.
4. F. Richen, "Modular Representations in Split BN Pairs," *Trans. Amer. Math. Soc.* **140** (1969), 435–460.
5. H. Wielandt, "Zum Satz von Sylow," *Math. Z.* **60** (1954), 407–408.

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