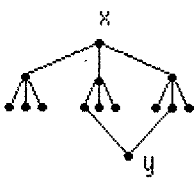
A brief survey of generalized polygons*

William M. Kantor
University of Oregon

Let $\Gamma=(V,E)$ be a finite connected graph (undirected, without loops and multiple edges). Let d be its diameter and g its girth (the size of a smallest circuit). Then $g\le 2d$ (trivially: see below). In this survey we will consider the case $g=2d$ for bipartite graphs: existence, properties and characterizations of such graphs.

Fix $x\in V$. In the picture



the treelike aspect can cease only if $d(x,y)\ge g/2$. Thus, $d\ge g/2$. Moreover, if $d=g/2$ then Γ is "locally treelike".

Example. $g=2d$ for an ordinary 2d-gon.

Definition (Tits). A generalized d-gon is a connected bipartite graph of diameter d and girth 2d in which all vertices have degree >2.

It is evident that generalized d-gons possess a certain amount of "combinatorial symmetry". Later we will discuss further symmetry imposed by the automorphism groups of certain of these graphs.

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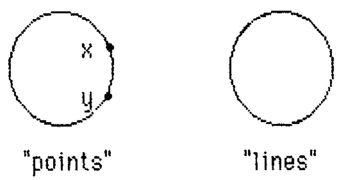
*The preparation of this paper was supported in part by NSF Grant DMS-8320149.

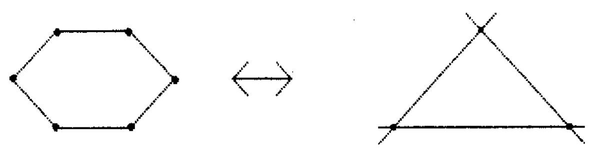
Example. d=2.

Here d(x,y) is odd and ≤d=2. Thus, Γ is a complete bipartite graph. The converse is obvious.

Example. d=3.

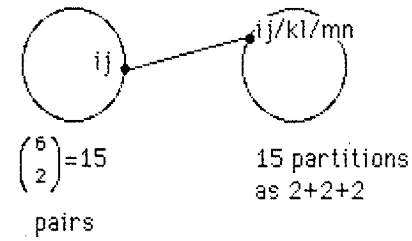


Call the members of the "halves" of Γ "points" and "lines". Since there are no circuits of length 4, two points are never both joined to two lines. Also, d(x,y) is even and ≤3. Thus, any two distinct points are joined to a unique line (and vice versa). It follows that generalized 3-gons are essentially the same as projective planes. Moreover, 6-gons in Γ correspond to triangles in the plane.



This translation from graph to point-line terminology occurs for all d≥3, and accounts for the name "generalized d-gon" -- in the sense that projective planes generalize ordinary triangles (3-gons).

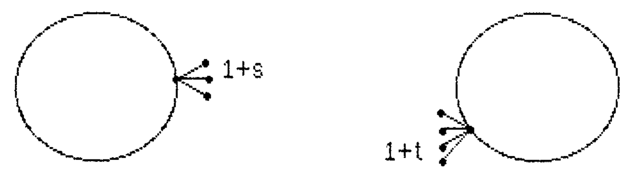
Example. d=4, all vertices of degree 3. Consider the set {1,2,3,4,5,6}.



This picture defines a bipartite graph Γ with 30 vertices. Clearly, $S_6 \le \mathrm{Aut}\,\Gamma$. In fact, Aut Γ = Aut S_6 has elements interchanging the two "halves".

The last two examples were regular. Complete bipartite graphs are not quite regular.

Proposition. Γ is left and right regular:



for constants s and t depending only on whether a point is in the left or right half.

When d=3, s=t is just the order of the projective plane. This is one of the reasons degrees are written in the above form ("1+s" rather than "k"). The proof of the proposition is not difficult, and relies on the fact that all degrees are >2.

The main restriction on generalized polygons is the

Feit–Higman Theorem [5]. d=2,3,4,6 or 8.

In addition, there are many restrictions on s and t, the foremost being as follows:

$d=6 \Rightarrow st$ is a square [5];

$d=8 \Rightarrow 2st$ is a square [5], so that Γ cannot be regular;

$d=4$ or $8 \Rightarrow t \le s^2$ and $s \le t^2$ [9];

$d=6 \Rightarrow t \le s^3$ and $s \le t^3$ [7].

The remaining types of restrictions when $d \ge 4$ are divisibility conditions satisfied by s and t. However, <u>there is no Bruck-Ryser type of theorem known when</u> $d \ge 4$.

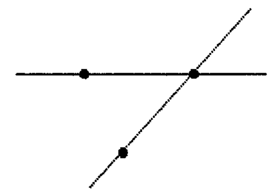
The list of <u>KNOWN</u> generalized d-gons ($d \ge 4$) is as follows (where q always denotes a power of a prime p).

d	s	t	KNOWN	Remarks
4	q	q	1 for each q; many for even q [4, p. 304;14]	regular graph
4	q+1	q−1	" [1,8,14]	
4	q	q^2	≥1 per q	$t=s^2$
			many if $q=p^e>p$ is odd [12]	
			≥2 per odd q>3 [10, 12; 6,18]	
			≥4 per $q=2^{2e+1}>2$ [4, p. 304; 10, 15]	
4	q^3	q^2	1 per q	
6	q	q	1 per q	regular graph
6	q	q^3	1 per q	$t=s^3$
8	q	q^2	1 per $q=2^{2e+1}$	$t=s^2$

Note that s and t need not be prime powers in the second row of the table. When equality holds in the inequality $t \le s^2$ or $t \le s^3$, further combinatorial regularity can be deduced [3,9]; but such regularity is reasonably well understood only in the case d=4.

We next turn to symmetry imposed by automorphism groups.

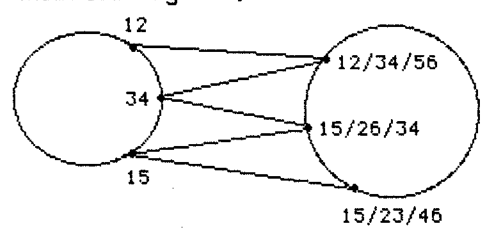
<u>Example</u>. d=3. The "best" projective planes are the desarguesian ones, in which Aut Γ is highly transitive. Namely, each such plane arises from a 3-dimensional vector space V, with points being 1-spaces, lines 2-spaces, and adjacence containment. Since GL(V) is transitive on the set of bases of V, it is transitive on the set of figures



in the plane. Also, $V \cong V^*$ (the dual space), so that Aut Γ interchanges points and lines. Thus, Aut Γ is transitive on the set of figures

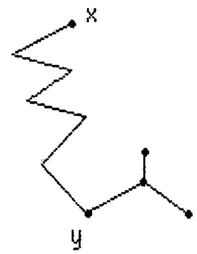
in the graph: <u>it is 4-arc transitive</u>.

<u>Example</u>. d=4, with 30 vertices as before. A typical 5-arc (path of length 5 without doubling back) is



It follows readily that this graph is 5-arc transitive.

<u>Digression</u>. How much arc-transitivity is allowed in any connected graph? Let d be the diameter, and let d(x,y)=d. The picture



shows that d+1-arc transitivity is the best one can hope for. (Compare [2, p. 113].)

<u>Snag</u>: In the case of generalized d-gons,

there exist 4-arc transitive generalized 3-gons, one per s=t=q,

there exist 5-arc transitive generalized 4-gons $\Leftrightarrow$ $s=t=2^e$ (one per e),

there exist 7-arc transitive generalized 6-gons $\Leftrightarrow$ $s=t=3^e$ (one per e).

Thus, one cannot expect too much arc transitivity without severe additional consequences. This phenomenon is already evident in the following classical result (which is <u>not</u> specifically concerned with generalized polygons).

<u>Theorem</u> (Tutte [19; 2, p. 124]). If Γ is a trivalent ℓ-arc transitive graph then $\ell \le 5$.

This theorem has been generalized as follows:

<u>Theorem</u> (Weiss [20]). If Γ is an ℓ-arc transitive graph that is not ℓ+1-arc transitive, and if its degree is $k \ge 3$, then the following all hold:

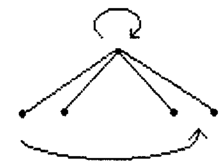
$\ell \le 5$ or $\ell = 7$;

if $\ell \ge 4$ then k−1 is a prime power;

if $\ell = 5$ then $k-1=2^e$ (for some e); and

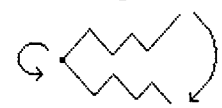
if $\ell = 7$ then $k-1=3^e$ (for some e).

Here, Γ need not be a generalized ℓ-1-gon. However, Weiss in a sense "embeds" a generalized ℓ-1-gon into Γ (which explains, to some extent, the restrictions occurring when ℓ=5 or 7). The main part of his proof uses the classification of finite simple groups in the following manner. If $\ell \ge 2$ then Aut Γ is transitive on 2-arcs. This implies that the stabilizer



of a vertex is 2-transitive on the set of adjacent vertices. Since all finite 2-transitive groups are now known, this provides the initial data for a clever argument.

The snag mentioned earlier concerning ℓ-arc transitivity of generalized polygons can be avoided by introducing the following extension. A <u>LOCALLY ℓ-arc transitive</u> graph is one in which, for each vertex x and each pair of ℓ-arcs starting at x, there is an automorphism



fixing x and sending the first ℓ-arc to the second one. However, now ℓ can be arbitrarily large, and hence d is no longer bounded. The study of special classes of locally ℓ-arc transitive graphs is an active research area in finite group theory.

The "nicest" locally ℓ-arc transitive graphs are generalized ℓ-1-gons. These have been characterized completely. For example, the only 4-arc transitive generalized 3-gons are the desarguesian projective planes. The complete list of <u>all</u> locally ℓ-arc transitive generalized ℓ-1-gons is as follows.

$d=\ell-1$	s	t	
3	q	q	one per q (=prime power)
4	q	q	"
4	q	q^2	"
4	q^3	q^2	"
6	q	q	"
6	q	q^3	"
8	q	q^2	one per $q=2^{2e+1}$

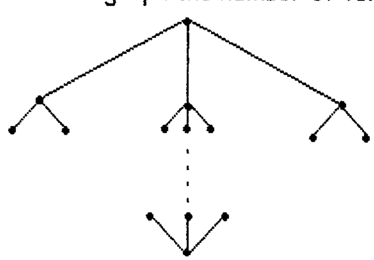
In each of the above generalized polygons, the full automorphism group is also the automorphism group of a finite simple group.

Conjecture. Every edge-transitive generalized d-gon ($d\geq 3$) is one of the above ones, with just two exceptions (due to Marshall Hall) having $d=4$, $s=3$, $t=5$ or $d=4$, $s=15$, $t=17$.

Other properties or occurrences of generalized polygons.

1. They occur as building blocks for tripartite, 4-partite,..., graphs related to finite simple groups [11,13]. This is a very active area of research for both geometers and group theorists.

2. They arise as extremal regular graphs of given degree $k>2$ and girth g. Namely, for such a graph the number of vertices is



$\geq 1+k+k(k-1)+\cdots+k(k-1)^{\frac{1}{2}g-2}+(k-1)^{\frac{1}{2}g-1}$. Equality holds iff the graph is a generalized $\frac{1}{2}g$-gon [2, p.154].

3. Tanner has used them to construct codes [16] and expanders [17].

They provide good expanders: for any set Y of vertices,

(# vertices joined to at least one member of Y)/|Y|

is unusually large (see [17] for a precise statement). While this may not seem surprising in view of the tree-like nature of these graphs, the proof is matrix-theoretic, involving the eigenvalues of the adjacency matrix (using information occurring in the proof of the Feit-Higman Theorem).

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