Cycles in Graphs and Groups

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1. INTRODUCTION. If a group G of automorphisms of a graph Γ acts transitively on the set of vertices, then Γ is d-valent for some d: each vertex is adjacent to exactly d others. This note concerns cycles in Γ, by which we will mean subgraphs isomorphic to a k-cycle for some $k \geq 3$; hence there will be no "initial" vertex. If $g \in G$ and C is a cycle in Γ, then $g(C)$ is another cycle in Γ: G acts on the set of all cycles in Γ.

There is another meaning of the word "cycle": each element of G can be written as a product of disjoint cycles of the set of vertices. If $C = (1, \dots, k)$ is such a cycle and $g \in G$, then an elementary calculation gives $gCg^{-1} = (g(1), \dots, g(k))$, and this is just $g(C)$ if C is viewed as a cyclically ordered set of vertices. In other words, the conjugation and functional actions behave the same.

In a 1971 lecture, J. H. Conway [**6**] presented the following result, which merges these two meanings of "cycle" when G can move any ordered pair of adjacent vertices to any other such pair:

Theorem 1 (Conway). *Let G be a group of automorphisms of a d-valent graph Γ ($d \geq 2$). Assume that G acts transitively on the set of all ordered pairs of adjacent vertices. Let X be the set of all (ordered) cycles in Γ each of which is* also *a cycle occurring in some element of G. Then G has exactly $d - 1$ orbits in its action on X.*

N. Biggs described this theorem in [**2**], with a brief sketch of Conway's proof; and also in [**1**, p. 75] and [**3**, p. 137]. More complete proofs have been provided in [**7, 8**]. In this note we give a short proof showing that this is essentially an elementary group-theoretic or combinatorial result. See the next section for examples.

The theorem seems remarkable for its generality, and for the precise number of orbits it contains. Studying this theorem led to the rediscovery of a more elementary result, due to R. Parker [**5**, p. 48], having a similar flavor:

Theorem 2 (Parker). *Let G be any subgroup of the symmetric group of degree n. Let X be the set of all cycles occurring in the elements of G. Then G has exactly n orbits in its action on X by conjugation.*

Of course, in general X is not a subset of G.

The next theorem allows us to relate the two previous ones: Theorem 2 can be viewed as a special case of Theorem 3, which we will see is only superficially more general than Theorem 1. The occurrences of the word "ordered" suggest that Theorem 1 is really about digraphs. Therefore, let G be a group of automorphisms of a digraph Γ; let $X(\Gamma, G)$ be the set of all (directed) cycles of Γ, including ones of length 2, each of which is also a cycle occurring in some element of G. Then G acts on $X(\Gamma, G)$.

Theorem 3. *If G is a vertex-transitive group of automorphisms of a digraph Γ with outdegree $d \geq 1$, then G has exactly d orbits on $X(\Gamma, G)$.*

We prove Theorems 1 and 3 in Section 3 using a general preliminary result from Section 2. In Section 4 we prove Theorem 2 and indicate how it relates to Theorem 3.

2. CYCLES. We consider the symmetric group S_n, acting on $\{1, 2, \dots, n\}$. We have seen that the action of S_n on cyclically ordered subsets of $\{1, 2, \dots, n\}$ is the same as the action by conjugation on cyclic permutations. We will view each cycle in S_n as a cyclically ordered set.

Let G be a subgroup of S_n, and let G_1 and G_{12} be the stabilizers in G of 1, and of both 1 and 2, respectively. Let $X_{12}(G)$ be the set of all cycles $(1, 2, \dots)$ occurring in those elements of G that move 1 to 2. Then G_{12} acts on $X_{12}(G)$.

Proposition 4. *If there is an element of G moving 1 to 2, then G_{12} has exactly $|G_1 : G_{12}|$ orbits on $X_{12}(G)$.*

Group-theoretic examples. Let G be the alternating group A_5 acting on $\{1, 2, 3, 4, 5\}$. When G_{12} acts on $X_{12}(G)$ by conjugation (via $h \to ghg^{-1}$, $g \in G_{12}$), there are $|G_1 : G_{12}| = 4$ orbits, with representatives $(1, 2)$, $(1, 2, 3)$, $(1, 2, 3, 4, 5)$ and $(1, 2, 3, 5, 4)$. The first of these cycles is not in G, but arises from the element $(1, 2)(3, 4)$ of G. On the other hand, if G is S_5 then 4 orbit representatives are $(1, 2)$, $(1, 2, 3)$, $(1, 2, 3, 4)$ and $(1, 2, 3, 4, 5)$. Yet another instance for $n = 5$ is the affine group $\{x \to ax + b \mid a, b \in \mathbb{Z}_5,\ a \neq 0\}$ acting on $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$, where $X_{12}(G)$ consists of the cycles $(1, 2, 3, 4, 0)$, $(1, 2, 4, 3)$, $(1, 2, 0, 4)$ and $(1, 2)$ (not in the group), arising from the permutations $x \to ax + 2 - a$ with $a = 1, 2, 3, 4$, respectively.

In Theorem 2 we must also include a single 1-cycle as an orbit representative for each of these examples.

Graph-theoretic examples. Each of the preceding three examples can be viewed in the setting of Theorem 1, in which Γ is the complete graph K_5 on 5 vertices. Each group is 2-transitive: it is transitive on the ordered pairs of adjacent vertices. This graph has valence $d = 4$. The cycle $(1, 2)$ of a permutation must be discarded since it is not a cycle of the graph.

Proof. The misnamed "Burnside's Lemma" (see [**10**, pp. 100–101], [**4**, p. 577] or [**9**]) concerns a group acting as a group of permutations of a finite set of points; it states that the average number of points fixed by a group element is the number of orbits of the group. In our setting the group is G_{12} and $X_{12}(G)$ is the set of "points":

$$\begin{aligned}|G_{12}|(\#G_{12}\text{-orbits on } X_{12}(G)) &= \sum_{g \in G_{12}} (\#C \in X_{12}(G) \text{ fixed by } g) \\ &= \sum_{C \in X_{12}(G)} (\#g \in G_{12} \text{ fixing } C) \\ &= \sum_{C \in X_{12}(G)} |G_{(C)}|,\end{aligned}$$

since any $g \in G_{12}$ fixing the cycle C must lie in the pointwise stabilizer $G_{(C)}$ of C. (Thus, $G_{(C)}$ is the set of all elements of G fixing every member of C. The second equality above is obtained by counting in two ways the number of pairs (g, C) with $g \in G_{12}$ and $C \in X_{12}(G)$ fixed by g.)

If g_C is one of the elements of G having C as a cycle, then $g_C G_{(C)}$ is the set of *all* such elements of G. (If f is any such element then $g_C^{-1} f$ fixes every member of C.)

Similarly, if $h \in G$ moves 1 to 2, then hG_1 is the set of *all* $g \in G$ sending 1 to 2. Each such g has a unique cycle $(1, 2, \dots) \in X_{12}(G)$. Thus,

$$\sum_{C \in X_{12}(G)} |G_{(C)}| = \sum_{C \in X_{12}(G)} |g_C G_{(C)}| = |hG_1| = |G_1|,$$

so that

$$(\#G_{12}\text{-orbits on } X_{12}(G)) = |G_1|/|G_{12}|. \qquad \blacksquare$$

3. GRAPHS AND DIGRAPHS.

Proof of Theorem 3. We have already observed that G acts on $X(\Gamma, G)$. Let $V = \{1, 2, \dots, n\}$ be the set of vertices of Γ.

We first note that the edge-transitive version of Theorem 3 implies the general version. For, let I denote a set of representatives of the orbits of G_1 on the vertices i adjacent from 1 (thus, $(1, i)$ is an edge and $G(1, i)$ is a set of edges). If $i \in I$ then $\Gamma_i = (V, G(1, i))$ is a digraph on which G acts as a group of automorphisms transitive on both the vertices and the edges. Since the edge-set of Γ is the disjoint union of the edge-sets of the digraphs Γ_i, if d_i is the outdegree of Γ_i then $d = \sum_{i \in I} d_i$. Moreover, $X(\Gamma, G)$ is the disjoint union of the sets $X(\Gamma_i, G)$: any cycle in $X(\Gamma, G)$ has all its edges in the same orbit $G(1, i)$ for a unique $i \in I$. Thus, if the theorem holds for each Γ_i then it holds for Γ.

Hence, we may assume that $I = \{2\}$ and $\Gamma = \Gamma_2$ (thus, we are now considering the digraph version of Theorem 1). Let G_1, G_{12} and $X_{12}(G)$ be as in Proposition 4. Note that $X_{12}(G) \subseteq X(\Gamma, G)$: if $g \in G$ moves 1 to 2 then any successive vertices in the cycle $(1, 2, \dots)$ of the permutation g are adjacent in Γ, so that this cycle of g is also a cycle of Γ. In particular, $X_{12}(G)$ consists of all members of $X(\Gamma, G)$ containing the edge $(1, 2)$. This observation provides the link between the purely group-theoretic set $X_{12}(G)$ and the graph-theoretic set $X(\Gamma, G)$.

Every G_{12}-orbit on $X_{12}(G)$ is contained in a unique G-orbit on $X(\Gamma, G)$.

On the other hand, since $I = \{2\}$ every G-orbit on $X(\Gamma, G)$ contains a cycle in $X_{12}(G)$. Suppose that $C = (1, 2, \dots)$ and $C' = (1, 2, \dots)$ are two members of $X_{12}(G)$ lying in the same G-orbit; we claim that they are in the same G_{12}-orbit. For, if $g(C') = C$ for some $g \in G$, and if C occurs as a cycle of the element $g_C \in G$, then some power g_C^k sends the edge $g(1, 2)$ of $g(C') = C$ to the edge $(1, 2)$ of C, so that $g_C^k g \in G_{12}$ sends C' to C.

Consequently, the number of G-orbits on $X(\Gamma, G)$ equals the number of G_{12}-orbits on $X_{12}(G)$, which is $|G_1 : G_{12}| = d$ by Proposition 4. $\blacksquare$

Theorem 1 easily follows from Theorem 3: the digraph Γ' required in Theorem 3 is obtained from the graph Γ in Theorem 1 by replacing each edge by two directed edges in opposite directions. Now delete the orbit of 2-cycles from $X(\Gamma', G)$ in order to deduce Theorem 1.

4. PERMUTATION GROUPS. In order to prove Theorem 2 we will restrict the previous results to the case of the complete graph K_n. Since every cycle of length at least 3 of any permutation of the vertices is automatically a cycle of K_n, this amounts to dealing with an arbitrary subgroup of the symmetric group on $\{1, \dots, n\}$, and then allowing cycles of any length; once again we will view cycles as cyclically ordered sets of points. This is exactly the situation in Theorem 2.

Proof of Theorem 2. As in the proof of Proposition 4, if X is as in Theorem 2 then

$$(\#G\text{-orbits on } X)|G| = \sum_{C\in X}(\#g \text{ fixing } C) = \sum_{C\in X}|C||G_{(C)}|,$$

since the pointwise stabilizer $G_{(C)}$ has index $|C|$ in the group of all elements of G fixing the ordered cycle C. If g_C is again an element in G having C as a cycle, then $g_C G_{(C)}$ is again the set of all such elements. Count in two ways the triples (g, i, C), where $g \in G$ and $C \in X$ is the cycle of g containing i:

$$\begin{aligned}|G|n &= \sum_{g\in G}\sum_{\substack{C\in X \text{ is a}\\ \text{cycle of } g}}|C| \\ &= \sum_{C\in X}|C|(\#g \in G \text{ having } C \text{ as a cycle}) \\ &= \sum_{C\in X}|C||g_C G_{(C)}|.\end{aligned}$$

Thus, $(\#G\text{-orbits on } X)|G| = n|G|$. ■

The preceding short proof of Theorem 2 is slightly different from the one in [**5**, p. 48], and is suspiciously similar to the proof of Proposition 4. In fact, Theorem 2 *is essentially a special case of* Theorem 3 *for* K_n (turned into a digraph K_n' as in the preceding section). For, in Theorem 2 it suffices to restrict to each orbit of G on $\{1, \dots, n\}$, and hence to assume that G is transitive on $\{1, \dots, n\}$. Then Theorem 3 states that G has $d = n - 1$ orbits on $X(K_n', G)$; this is essentially the set X in Theorem 2 with the orbit of 1-cycles deleted.

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