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Envelopes of Geometric Lattices*

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A categorical embedding theorem is proved for geometric lattices. This states roughly that, if one wants to consider only those embeddings into projective spaces having a suitable universal property, then the existence of such an embedding can be checked by seeing whether corresponding properties hold for many small intervals. Tutte's embedding theorem for binary geometric lattices is a consequence of this result.

1. INTRODUCTION

The purpose of this paper is to prove a somewhat technical theorem which provides a sufficient condition in order that a geometric lattice **G** be embeddable in a finite-dimensional projective space over a given field K. The condition involves all the small dimensional intervals of **G** containing 1. Instead of considering arbitrary embeddings, we restrict our attention to those having a universal property which implies that all embeddings of **G** in a finite-dimensional projective K-space are essentially the same. In order to state the theorem, we require some terminology.

DEFINITIONS. 1. The *dimension* dim X of an element X of a geometric lattice is one less than the size of a basis of X. (This is the concept of dimension used in projective geometry: planes are 2-dimensional.)

2. An *isometry* from a geometric lattice **G** to a geometric lattice **H** is an order-monomorphism $i: \mathbf{G} \to \mathbf{H}$, mapping 1 to 1, preserving the dimension of each element of dimension at most $\max(1, \dim \mathbf{G} - 1)$, and such that $i(X \vee Y) = i(X) \vee i(Y)$ whenever $X, Y \in G$ and $X \vee Y \neq 1$. (For example, the inclusion map is an isometry from the 2-dimensional lattice of points and lines of $PG(n, K)$ into $PG(n, K)$, where $n \geqslant 2$. This definition thus properly contains the corresponding one in [3].)

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3. If K is a field and $\mathbf{G}$ a geometric lattice, a *K-envelope* of $\mathbf{G}$ is a pair $(\mathbf{E}(\mathbf{G}), i)$ consisting of a finite-dimensional projective K-space $\mathbf{E}(\mathbf{G})$ and an isometry $i\colon \mathbf{G} \to \mathbf{E}(\mathbf{G})$ such that, whenever $\alpha\colon \mathbf{G} \to \mathbf{M}$ is an isometry from $\mathbf{G}$ into a finite-dimensional projective K-space $\mathbf{M}$, there is a unique isometry $\beta\colon \mathbf{E}(\mathbf{G}) \to \mathbf{M}$ such that

$$\begin{tikzcd}
\mathbf{G} \arrow[rr] \arrow[dr, "\alpha"'] & & \mathbf{E}(\mathbf{G}) \arrow[dl, "\beta"] \\
& \mathbf{M} &
\end{tikzcd}$$

is commutative. (Note that $\dim \mathbf{G}$ and $\dim \mathbf{E}(\mathbf{G})$ are, in general, different.)

4. $\mathbf{G}$ is *K-rigid* if, for each isometry $\alpha\colon \mathbf{G} \to \mathbf{M}$ from $\mathbf{G}$ into a finite-dimensional projective K-space $\mathbf{M}$ spanned by $\alpha(\mathbf{G}) - \{1\}$, the identity is the only collineation of $\mathbf{M}$ inducing the identity on $\alpha(\mathbf{G})$. (The existence of a K-envelope implies K-rigidity if $\dim \mathbf{G} \geqslant 2$ and K is isomorphic to no proper subfield of itself.)

5. For $X \in \mathbf{G}$, $\mathbf{G}^X$ denotes the interval $[X, 1]$. Here $\dim \mathbf{G}^X = \dim \mathbf{G} - \dim X - 1$.

The following is our main result:

THEOREM 1. *Let K be a field isomorphic to no proper subfield of itself, $j \geqslant 1$ an integer, and $\mathbf{G}$ a geometric lattice with $\dim \mathbf{G} \geqslant j + 2$. Assume that, for all $W \in \mathbf{G}$,*

(a) *$\mathbf{G}^W$ has a K-envelope whenever $\dim \mathbf{G}^W = j + 1$ or $j + 2$,*

and

(b) *$\mathbf{G}^W$ is K-rigid whenever $\dim \mathbf{G}^W = j$.*

Then $\mathbf{G}$ has a K-envelope.

From a geometric point of view, the most interesting case of this theorem is when $j = 1$. In this case, our definition of isometries requires that $\dim \mathbf{E}(\mathbf{G}) = \dim \mathbf{G}$. Also, when $j = 1$ and $K \neq GF(2)$, each element of dimension $\dim \mathbf{G} - 2$ is on at least 3 hyperplanes—in fact, at least 4 hyperplanes if K is not a prime field. When $j \geqslant 2$, there are no such restrictions on hyperplanes or $\dim \mathbf{E}(\mathbf{G})$.

The main theorems of [3] are similar types of embedding theorems. There are two ideas in their proofs. One of these is generalized in the embedding lemma of [3]; the other—using induction and gluing together modular lattices—is generalized in the present paper. However, here the glue used is categorical instead of geometric.

The results of [3] are not contained in Theorem 1. Thus, the main theorems there are concerned with embedding a geometric lattice **G** into a modular geometric lattice—not necessarily a projective space—of the same dimension as **G**. Also, in [3] there are no restrictions placed on the fields involved. However, the more useful special cases studied there are contained in Theorem 1. For example, Theorem 1 implies the (well-known) existence of a 6-dimensional projective $GF(3)$-representation of the Mathieu group M_{12}. It also implies the other results on extensions of finite inversive planes contained in [3, Section 5]. In addition, it provides a new proof of Tutte's theorem that binary matroids are representable over $GF(2)$.

While the notion of a K-envelope is natural, it is, unifortunately, stronger than one would like. For example, a triangle has no K-envelope for $K \neq GF(2)$, as there always exist nontrivial collineations of $PG(2, K)$ fixing a triangle pointwise. Another deficiency of K-envelopes is that **G** can have one and some $\mathbf{G}^p$ not (where p is a point).

Finally, the field K must be specified throughout this paper because of the following type of example: $AG(2, 3)$ is contained in $PG(2, 4)$, but of course $PG(2, 3)$ is not.

2. Preliminary Results

G will always denote a (not necessarily finite) geometric lattice. We write $\mathbf{G}^{\#} = \mathbf{G} - \{1\}$. The element spanned by a subset S of **G** is the join of S. p, q, r will always denote points of G. Isomorphisms will be bijective. Most other definitions were already given in Section 1. All other terms can be found in [2], [3], or [4].

K will denote a (not necessarily commutative) field. Projective K-spaces will always be finite dimensional.

The following obvious fact will be used frequently: if S spans **G**, and $p \in S$, then the set of elements $p \vee X$, $X \in S$, spans $\mathbf{G}^p$.

Note that, by definition, the composition of two isometries is again an isometry.

Lemma 1. *Let* **G** *and* **H** *be geometric lattices and* i: $\mathbf{G} \to \mathbf{H}$ *an isometry. Then*

(a) $i(\mathbf{G})$ *has a natural lattice structure such that* i: $\mathbf{G} \to i(\mathbf{G})$ *is an isomorphism*;

(b) *if* $X, Y \in \mathbf{G}$ *is a modular pair with* $X \vee Y \neq 1$, *then* $i(X \wedge Y) = i(X) \wedge i(Y)$; *and*

(c) $\dim \mathbf{G} \leqslant \dim \mathbf{H}$.

Proof. (a) is obvious. In (b), $i(X \wedge Y) \leqslant i(X) \wedge i(Y)$, while

$$\begin{aligned}\dim i(X \wedge Y) &= \dim X \wedge Y = \dim X + \dim Y - \dim X \vee Y \\ &= \dim i(X) + \dim i(Y) - \dim i(X) \vee i(Y) \\ &\geqslant \dim i(X) \wedge i(Y),\end{aligned}$$

so $i(X \wedge Y) = i(X) \wedge i(Y)$. This proves (b). Since i maps chains of $\mathbf{G}$ to chains of $\mathbf{H}$, (c) is clear.

LEMMA 2. *Suppose K is isomorphic to no proper subfield of itself. Let $i\colon \mathbf{M} \to \mathbf{N}$ be an isometry of projective K-spaces of dimension $\geqslant 2$. If $i(\mathbf{M})^{\#}$ spans* $\mathbf{N}$, *then* $i(\mathbf{M}) = \mathbf{N}$ *and i is an isomorphism.*

Proof. First suppose $\dim \mathbf{M} = 2$. If $X, Y \in \mathbf{M}$ are lines, there is a point p on X and Y. Then $i(p)$ is on $i(X)$ and $i(Y)$. Thus the lines in $i(\mathbf{M})$ are pairwise coplanar, but are not concurrent, and hence all lie in a plane of $\mathbf{N}$. Since $i(\mathbf{M})^{\#}$ spans $\mathbf{N}$, $\dim \mathbf{N} = 2$, and $i(\mathbf{M})$ is a subplane of $\mathbf{N}$ isomorphic to $\mathbf{N}$. The restriction on K now implies that $i(\mathbf{M}) = \mathbf{N}$.

Now let $\dim \mathbf{M} \geqslant 3$. Then the restriction of i to any plane E of $\mathbf{M}$ induces an isomorphism from E into the plane $i(E)$. Hence, each point and line of $i(E)$ is in $i(\mathbf{M})$, so the points of $i(\mathbf{M})$ consist of all points of some subspace X of $\mathbf{N}$. Since $i(\mathbf{M})^{\#}$ spans $\mathbf{N}$ and $X \geqslant Y$ for all $Y \in i(\mathbf{M})^{\#}$, we must have $X = 1$.

(Obviously, Lemma 2 holds for $\dim \mathbf{M} = 1$ if and only if K is finite.)

LEMMA 3. *If* $(\mathbf{E}(\mathbf{G}), i)$ *is a K-envelope of* $\mathbf{G}$, *then* $i(\mathbf{G})^{\#}$ *spans* $\mathbf{E}(\mathbf{G})$.

Proof. Set $M = \vee i(\mathbf{G})^{\#}$, so $\mathbf{M} = [0, M]$ is a projective K-space. Suppose $M \neq 1$. Then $M \notin i(\mathbf{G})^{\#}$, as otherwise $1 \neq i^{-1}(M) \geqslant X$ for all $X \in \mathbf{G}^{\#}$. Define $i'\colon \mathbf{G} \to \mathbf{M}$ to be i on $\mathbf{G}^{\#}$, and let $i'(1) = M$. Then i' is an isometry. There is an isomorphism $\beta\colon \mathbf{E}(\mathbf{G}) \to \mathbf{M}$ with $\beta i = i'$. By Lemma 1(c), $\dim \mathbf{E}(\mathbf{G}) \leqslant \dim \mathbf{M}$, which is not the case.

LEMMA 4. *Suppose K is isomorphic to no proper subfield of itself, and* $\dim \mathbf{G} \geqslant 3$. *Let p be a point of* $\mathbf{G}$, $(\mathbf{E}(\mathbf{G}), i)$ *a K-envelope of* $\mathbf{G}$, *and* $(\mathbf{E}(\mathbf{G}^p), i^p)$ *a K-envelope of* $\mathbf{G}^p$. *Then*

(a) $\dim \mathbf{E}(\mathbf{G}^p) = \dim \mathbf{E}(\mathbf{G}) - 1$, *and*

(b) *if* $\varphi\colon \mathbf{E}(\mathbf{G}^p) \to \mathbf{E}(\mathbf{G})$ *is the isometry making*

$$\begin{tikzcd} \mathbf{G} \arrow[r, "i"] & \mathbf{E}(\mathbf{G}) \\ \mathbf{G}^p \arrow[u, phantom, "\cup"] \arrow[r, "i^p"] & \mathbf{E}(\mathbf{G}^p) \arrow[u, "\varphi"'] \end{tikzcd}$$

commutative, then φ is an isomorphism from $\mathbf{E}(\mathbf{G}^p)$ *onto* $\mathbf{E}(\mathbf{G})^{i(p)}$.

Proof. Since $i^p(p)$ is the 0-element of $\mathbf{E}(\mathbf{G}^p)$, $X \in \mathbf{E}(\mathbf{G}^p)$ implies $X \geqslant i^p(p)$ and hence $\varphi(X) \geqslant \varphi i^p(p) = i(p)$. Thus, φ maps $\mathbf{E}(\mathbf{G}^p)$ into $\mathbf{E}(\mathbf{G})^{i(p)}$.

By Lemma 3, $i(\mathbf{G})^{\#}$ spans $\mathbf{E}(\mathbf{G})$, so $(i(\mathbf{G})^{i(p)})^{\#}$ spans $\mathbf{E}(\mathbf{G})^{i(p)}$. (For, each element of $\mathbf{E}(\mathbf{G}^p)$ is a join of points $i(q)$, and hence of lines $i(p) \vee i(q) = i(p \vee q)$.) Since $i(\mathbf{G})^{i(p)} = i(\mathbf{G}^p) = \varphi i^p(\mathbf{G}^p)$, $(\operatorname{Im} \varphi)^{\#}$ spans $\mathbf{E}(\mathbf{G})^{i(p)}$. By Lemma 2, $\operatorname{Im} \varphi = \mathbf{E}(\mathbf{G})^{i(p)}$, so Lemma 4 holds.

LEMMA 5. *Suppose K is isomorphic to no proper subfield of itself, and* $\dim \mathbf{G} \geqslant 2$. *Then G has a K-envelope if and only if*

(a) *there is an isometry* $\alpha: \mathbf{G} \to \mathbf{M}$ *with* $\mathbf{M}$ *a projective K-space; and*

(b) *for each such* α *and* $\mathbf{M}$ *for which* $\alpha(\mathbf{G})^{\#}$ *spans* $\mathbf{M}$, *each isometry* $\alpha(\mathbf{G}) \to \mathbf{M}$ *is induced by a unique collineation of* $\mathbf{M}$.

Proof. Suppose $(\mathbf{E}(\mathbf{G}), i)$ is a K-envelope of $\mathbf{G}$. Then (a) holds. Suppose $\alpha(\mathbf{G})^{\#}$ spans $\mathbf{M}$. Let $\beta: \mathbf{E}(\mathbf{G}) \to \mathbf{M}$ be the isometry for which $\beta i = \alpha$. Then $\operatorname{Im} \alpha = \operatorname{Im} \beta i \subseteq \operatorname{Im} \beta$ implies that $(\operatorname{Im} \beta)^{\#}$ spans $\mathbf{M}$, so β is an isomorphism (by Lemma 2) and $\dim \mathbf{E}(\mathbf{G}) = \dim \mathbf{M}$. Now let $\gamma: \alpha(\mathbf{G}) \to \mathbf{M}$ be an isometry, and let $\delta: \mathbf{E}(\mathbf{G}) \to \mathbf{M}$ be the unique isometry for which $\gamma\alpha = \delta i$. Then δ is an isomorphism (by Lemma 2, since $\dim \mathbf{E}(\mathbf{G}) = \dim \mathbf{M}$), and $\gamma\alpha = \delta\beta^{-1}\alpha$. Consequently, $\delta\beta^{-1}$ is a collineation of $\mathbf{M}$ which agrees with γ on $\alpha(\mathbf{G})$. Also, if ϵ is another such collineation of $\mathbf{M}$, then $\gamma\alpha = \epsilon\alpha$, so $\epsilon\beta i = \epsilon\alpha = \gamma\alpha$. The uniqueness of δ now forces $\epsilon\beta = \delta$.

Conversely, assume (a) and (b). In (a) we may assume that $\alpha(\mathbf{G})^{\#}$ spans $\mathbf{M}$ and $\dim \mathbf{M}$ is minimal. We then claim that $(\mathbf{M}, \alpha)$ is a K-envelope of $\mathbf{G}$.

Suppose $\beta: \mathbf{G} \to \mathbf{N}$ is an isometry, where $\mathbf{N}$ is a projective K-space. We may assume that $\beta(\mathbf{G})^{\#}$ spans $\mathbf{N}$. By our choice of $\mathbf{M}$, there is an isometry $\sigma: \mathbf{M} \to \mathbf{N}$. Now note that $\sigma\alpha\beta^{-1}: \beta(\mathbf{G}) \to \mathbf{N}$ is an isometry. By (b), there is a unique collineation $\gamma: \mathbf{N} \to \mathbf{N}$ extending it. Then $\sigma\alpha\beta^{-1}\beta = \gamma\beta$, so $\beta = \gamma^{-1}\sigma\alpha$. Moreover, $\operatorname{Im} \beta \subseteq \operatorname{Im} \gamma^{-1}\sigma$. Since $\beta(\mathbf{G})^{\#}$ spans $\mathbf{N}$, so does $(\operatorname{Im} \gamma^{-1}\sigma)^{\#}$. Thus, $\gamma^{-1}\sigma: \mathbf{M} \to \mathbf{N}$ is an isomorphism (Lemma 2).

Finally, suppose $\epsilon: \mathbf{M} \to \mathbf{N}$ is an isometry such that $\beta = \epsilon\alpha$. Then ϵ is an isomorphism (Lemma 2), and $\epsilon^{-1}\gamma^{-1}\sigma\alpha = \alpha$. Now $\epsilon^{-1}\gamma^{-1}\sigma: \mathbf{M} \to \mathbf{M}$ is a collineation agreeing with the identity on $\alpha(\mathbf{G})$. By (b), $\epsilon^{-1}\gamma^{-1}\sigma = 1$, so ϵ is unique. This proves both Lemma 5 and the following fact.

COROLLARY. *Suppose K has no proper subfield isomorphic to itself and that* $(\mathbf{E}(\mathbf{G}), i)$ *is a K-envelope of* $\mathbf{G}$, *where* $\dim \mathbf{G} \geqslant 2$. *If* $\alpha: \mathbf{G} \to \mathbf{M}$ *is an isometry, where* $\mathbf{M}$ *is a projective K-space spanned by* $\alpha(\mathbf{G})^{\#}$, *then* $\dim \mathbf{E}(\mathbf{G}) = \dim \mathbf{M}$ *and* $(\mathbf{M}, \alpha)$ *is also a K-envelope of* $\mathbf{G}$.

LEMMA 6. *Suppose K has no proper subfield isomorphic to itself. If* $\dim \mathbf{G} \geqslant 2$ *and* $\mathbf{G}$ *has a K-envelope, then* $\mathbf{G}$ *is K-rigid.*

Proof. Lemma 5b.

The preceding lemma requires the assumption made on K. For example, let $\mathbf{M}$ be a finite-dimensional projective K-space isomorphic to a proper sublattice $\mathbf{G}$ of itself. Let $i: \mathbf{G} \to \mathbf{G}$ be the identity map. Then $(\mathbf{G}, i)$ is a K-envelope of $\mathbf{G}$. Let $\alpha: \mathbf{G} \to \mathbf{M}$ be the inclusion map. Then $\mathbf{M}$ can have nontrivial collineations inducing the identity on $\mathbf{G} = \alpha(\mathbf{G})$.

LEMMA 7. *If* $\mathbf{G}$ *is indecomposable, it is K-rigid whenever* $\mathrm{Aut}(K) = 1$ *(e.g., when K is a prime field or the real field).*

Proof. Let $i: \mathbf{G} \to \mathbf{M}$ be an isometry, where $i(\mathbf{G})^{\#}$ spans $\mathbf{M}$. Let φ be a collineation of $\mathbf{M}$ inducing the identity on $i(\mathbf{G})$. Then φ is induced by a linear transformation (since $\mathrm{Aut}(K) = 1$) and is the identity on a spanning set of points of $\mathbf{M}$. Thus φ is induced by a diagonalizable linear transformation, whose eigenspaces yield a direct decomposition of $i(\mathbf{G})$ unless $\varphi = 1$.

THEOREM 2. *Let K be a field not isomorphic to a proper subfield of itself,* $\mathbf{G}$ *a geometric lattice of dimension* $\geqslant 3$, *and* $\mathbf{M}$ *a projective K-space. Let p, q, r be a triangle of* $\mathbf{G}$. *Assume*:

(a) *there is an isometry* $i: \mathbf{G} \to \mathbf{M}$;

(b) $\mathbf{G}^p$, $\mathbf{G}^q$, *and* $\mathbf{G}^{p \vee q}$ *have K-envelopes*; *and*

(c) *either $p \vee q$ has just 2 points, or* $\mathbf{G}^r$, $\mathbf{G}^{p \vee r}$ *and* $\mathbf{G}^{q \vee r}$ *have K-envelopes.*

Then $\mathbf{G}$ *has a K-envelope.*

Proof. We will apply Lemma 5. Thus, we must show that, for each such $i: \mathbf{G} \to \mathbf{M}$ such that $i(\mathbf{G})^{\#}$ spans $\mathbf{M}$, each isometry $\alpha: i(\mathbf{G}) \to \mathbf{M}$ extends to a unique collineation of $\mathbf{M}$. Clearly, we may assume that $\mathbf{G} = i(\mathbf{G})$ and i is the inclusion map. Thus, if $X, Y \in G$ and $X \vee Y \neq 1$, then $X \vee Y$ is the same in $\mathbf{G}$ and $\mathbf{M}$, as is $X \wedge Y$ if X, Y is a modular pair (Lemma 1b).

We must show that any isometry $\alpha: \mathbf{G} \to \mathbf{M}$ extends to a unique collineation of $\mathbf{M}$. Clearly, α can be replaced by $\varphi\alpha$ for any collineation φ of $\mathbf{M}$. The main step of the proof consists of proving (*): for some such φ, the isometry $\beta = \varphi\alpha: \mathbf{G} \to \mathbf{M}$ fixes p, q, r, and induces the identity on $\mathbf{G}^p$ and $\mathbf{G}^q$.

There is a collineation φ_1 of $\mathbf{M}$ such that $\varphi_1\alpha(p) = p$. We may thus assume that $\alpha(p) = p$. By Lemma 5, there is a collineation φ_2 of $\mathbf{M}$ such that α and φ_2 agree on $\mathbf{G}^p$. Replace α by $\varphi_2^{-1}\alpha$, so $\alpha(p) = p$ and α induces the identity on $\mathbf{G}^p$.

In particular, $\alpha(p \vee q \vee r) = p \vee q \vee r$, so $\alpha(q \vee r)$ is a line of the plane $p \vee q \vee r$. Consequently, there is a perspectivity ([2, p. 30]) φ_3 of $\mathbf{M}$ with center p such that $\varphi_3\alpha(q \vee r) = q \vee r$. Replacing α by $\varphi_3\alpha$, we find that $\alpha(p) = p$, $\alpha(q \vee r) = q \vee r$, and α induces the identity on $\mathbf{G}^p$. Since $p \vee q$, $q \vee r$ is a modular pair of lines fixed by α, by Lemma 1b $\alpha(q) = q$. Similarly, $\alpha(r) = r$.

The restriction α^q of α to $\mathbf{G}^q$ extends to a collineation σ of $\mathbf{M}^q$. Here, σ induces the identity on $\mathbf{G}^{p\vee q}$, and hence on $\mathbf{M}^{p\vee q}$ (Lemma 5). Consequently, σ is a perspectivity of $\mathbf{M}^q$, and $q \vee p$ is a center. Let A be an axis of σ, so $A > q$ is a hyperplane of $\mathbf{M}$.

We are trying to prove (*). If $\sigma = 1$, then α induces the identity on $\mathbf{G}^p$ and $\mathbf{G}^q$. Suppose, therefore, that $\sigma \neq 1$. Then A is on every line $> q$ of $\mathbf{M}$ fixed by σ, except possibly for the center $q \vee p$ of σ. In particular, $A > q \vee r$.

Since $\mathbf{G}^{\#}$ spans $\mathbf{M}$, there is a point $t \nless A$ of $\mathbf{G}$. If possible, choose $t \neq p$. Clearly, $\alpha(t) < \alpha(p \vee t) = p \vee t$. There is thus a (p, A)-perspectivity φ_4 of $\mathbf{M}$ such that $\varphi_4\alpha(t) = t$. Since $A > q \vee r$, $\beta = \varphi_4\alpha$ fixes p, q, r, t, and induces the identity on $\mathbf{G}^p$. Moreover, the restriction β^q of β to $\mathbf{G}^q$ still extends to a $(q \vee p, A)$-perspectivity τ^q of $\mathbf{M}^q$. Once again, (*) holds if $\tau^q = 1$, so we assume $\tau^q \neq 1$. Since β^q fixes $q \vee t \nless A$, $q \vee t$ must be the center $q \vee p$ of τ^q. Also, if $x < A$ is a point of $\mathbf{G}$, then $p \vee x$, $q \vee x$ is a modular pair, where $p \vee x$ and $q \vee x$ are fixed by β (since $q \vee x < A$ is fixed by τ^q), so $\beta(x) = x$. Consequently, if $t = p$ then p is the only point of $\mathbf{G}$ not on A, so β is induced by the identity of $\mathbf{M}$. We may thus assume that $q \vee p = t \vee p$ has at least 3 points.

Now consider the restriction β^r of β to $\mathbf{G}^r$, the extension τ^r of β^r to $\mathbf{M}^r$, and the restrictions τ^{qr} and τ^{rq} of τ^q and τ^r to $\mathbf{M}^{q\vee r}$. By (c) and Lemma 5, τ^r exists, and $\tau^{qr} = \tau^{rq}$ since both extend the restriction of β to $\mathbf{G}^{q\vee r}$. Here, τ^{qr} is a perspectivity of $\mathbf{M}^{q\vee r}$ with center $q \vee r \vee p$ and axis A; moreover, $\tau^{qr} \neq 1$ since $\tau^q \neq 1$ and $q \vee r$ is not the center of τ^q. But τ^r is also a perspectivity, with center $r \vee p$, fixing $r \vee t \neq r \vee p$, and inducing $\tau^{rq} = \tau^{qr} \neq 1$ on $\mathbf{M}^{r\vee q}$. Thus, τ^r has axis A, center $r \vee p$, and fixes $r \vee t$, where $r \vee p \neq r \vee t \nless A$. This is impossible, since $\tau^r \neq 1$. Thus, if $t \neq p$, then necessarily $\tau^q = 1$, and hence β^q induces the identity on $\mathbf{G}^q$.

This proves (*). As before, if $x \nless p \vee q$ is a point of $\mathbf{G}$, then $\beta(p \vee x) = p \vee x$ and $\beta(q \vee x) = q \vee x$ imply, by Lemma 1b, that $\beta(x) = x$. If $p \vee q$ has just 2 points, it follows that $\beta = \varphi\alpha$ extends to the identity collineation of $\mathbf{M}$. Suppose $p \vee q$ has more than 2 points. Then β^r is induced by τ^r again. Restricting to $\mathbf{G}^{r\vee q}$, we find that τ^{rq} induces the identity on $\mathbf{G}^{r\vee q}$ and hence is the identity on $\mathbf{M}^{r\vee q}$. Thus, τ^r is a perspectivity, with $r \vee q$ and $r \vee p$ centers, so $\tau^r = 1$. Then β^r induces the identity on $\mathbf{G}^r$. It follows that $\beta(x) = x$ if $x < p \vee q$, so once again β extends to the identity of $\mathbf{M}$.

Finally, suppose γ is a collineation of $\mathbf{M}$ extending β. Then γ^p and γ^q extend β^p and β^q, and hence are 1 (Lemma 5). Consequently, γ is a perspectivity with centers p and q, so $\gamma = 1$. This proves the theorem.

3. The Induction Step

Theorem 1 is an easy consequence of the following result.

Main Lemma. *Let* $\mathbf{G}$ *be a geometric lattice with* $\dim \mathbf{G} \geqslant 4$ *and* K *be a field not isomorphic to a proper subfield of itself. Assume that, for all points* p, q, r, $\mathbf{G}^{p\vee q}$ *has a* K*-envelope and* $\mathbf{G}^{p\vee q\vee r}$ *is* K*-rigid. Then* $\mathbf{G}$ *has a* K*-envelope.*

Proof. The proof is broken into several steps.

(i) For each p, q let $(\mathbf{E}(\mathbf{G}^{p\vee q}), i^{p\vee q})$ be a K-envelope of $\mathbf{G}^{p\vee q}$. In particular, we will have $\mathbf{E}(\mathbf{G}^p)$ and i^p. There is a unique isometry φ_{pq} making

$$\begin{tikzcd} \mathbf{G}^p \arrow[r, "i^p"] & \mathbf{E}(\mathbf{G}^p) \\ \mathbf{G}^{p\vee q} \arrow[u, phantom, "\bigcup"] \arrow[r, "i^{p\vee q}"] & \mathbf{E}(\mathbf{G}^{p\vee q}) \arrow[u, "\varphi_{pq}"'] \end{tikzcd} \tag{1}$$

commutative. Clearly, $\varphi_{pp} = 1$. By Lemma 4,

$$\operatorname{Im} \varphi_{pq} = \mathbf{E}(\mathbf{G}^p)^{i^p(p\vee q)} \tag{2}$$

and $\dim \mathbf{E}(\mathbf{G}^p) = \dim \mathbf{E}(\mathbf{G}^{q\vee r}) + 1$ for all p, q, r with $q \neq r$.

We may assume that the $\mathbf{E}(\mathbf{G}^p)$ are pairwise disjoint. Let Σ be their union. Define a relation $\sim$ on Σ by: for $X, Y \in \Sigma$,

$$\begin{aligned} X \sim Y \Leftrightarrow \exists p, q\colon X \in \mathbf{E}(\mathbf{G}^p),\quad Y \in \mathbf{E}(\mathbf{G}^q),\quad X \geqslant i^p(p \vee q),\\ Y \geqslant i^q(p \vee q),\quad \text{and} \quad \varphi_{pq}^{-1}(X) = \varphi_{qp}^{-1}(Y). \end{aligned} \tag{3}$$

Note that $X \geqslant i^p(p \vee q)$ implies, by (2), that $X \in \operatorname{Im} \varphi_{pq}$.

(ii) The crucial step of the proof consists of proving that $\sim$ *is an equivalence relation.* It is clearly reflexive and symmetric. Suppose $X, Y, Z \in \Sigma$ and $Y \sim X \sim Z$. Let $X \in \mathbf{E}(\mathbf{G}^p)$, $Y \in \mathbf{E}(\mathbf{G}^q)$, $Z \in \mathbf{E}(\mathbf{G}^r)$. First note that

$$Y \geqslant i^q(p \vee q \vee r) \qquad \text{and} \qquad Z \geqslant i^r(p \vee q \vee r). \tag{4}$$

In fact, $X \geqslant i^p(p \vee q)$ and $i^p(p \vee r)$, so $X \geqslant i^p(p \vee q \vee r) \geqslant i^p(p \vee q)$. Then $\varphi_{qp}^{-1} Y = \varphi_{pq}^{-1} X \geqslant \varphi_{pq}^{-1} i^p(p \vee q \vee r) = i^{p\vee q}(p \vee q \vee r)$ by (1), so

$Y \geqslant \varphi_{qp} i^{p\vee q}(p \vee q \vee r) = i^q(p \vee q \vee r)$, again by (1). The other half of (4) follows by symmetry.

Next note that, by (2), $\varphi_{qp}\varphi_{pq}^{-1}(U)$ is defined for each $U \in \mathbf{E}(\mathbf{G}^p)^{i^p(p\vee q)}$, and is in $\mathbf{E}(\mathbf{G}^q)^{i^q(p\vee q)}$. Moreover, for each $S \in \mathbf{G}^{p\vee q}$, (1) and (2) imply that

$$\varphi_{qp}\varphi_{pq}^{-1} i^p(S) = \varphi_{qp} i^{p\vee q}(S) = i^q(S). \tag{5}$$

Now take $U \in \mathbf{E}(\mathbf{G}^p)^{i^p(p\vee q\vee r)}$. Then, by (2), $\varphi_{pr}\varphi_{rp}^{-1}\varphi_{rq}\varphi_{qr}^{-1}\varphi_{qp}\varphi_{pq}^{-1}(U)$ is defined, and is again in $\mathbf{E}(\mathbf{G}^p)^{i^p(p\vee q\vee r)}$. This defines a collineation ψ of the projective space $\mathbf{E}(\mathbf{G}^p)^{i^p(p\vee q\vee r)}$. Three applications of (5) show that, whenever $S \in \mathbf{G}^{p\vee q\vee r}$, $\psi i^p(S) = i^p(S)$. However, i^p induces an isometry of $\mathbf{G}^{p\vee q\vee r}$ into $\mathbf{E}(\mathbf{G}^p)^{i^p(p\vee q\vee r)}$, and $i^p(\mathbf{G}^{p\vee q\vee r})^{\#} = (i^p(\mathbf{G}^p)^{i^p(p\vee q\vee r)})^{\#}$ spans $\mathbf{E}(\mathbf{G}^p)^{i^p(p\vee q\vee r)}$. Since we are assuming that $\mathbf{G}^{p\vee q\vee r}$ is K-rigid, it follows that $\psi = 1$.

In particular, $\varphi_{pr}\varphi_{rp}^{-1}\varphi_{rq}\varphi_{qr}^{-1}\varphi_{qp}\varphi_{pq}^{-1}(X) = X$. Since $Y \sim X \sim Z$, $\varphi_{pq}^{-1}(X) = \varphi_{qp}^{-1}(Y)$ and $\varphi_{pr}^{-1}(X) = \varphi_{rp}^{-1}(Z)$. These equations unravel to give $\varphi_{qr}^{-1}(Y) = \varphi_{rq}^{-1}(Z)$, where $Y \geqslant i^q(q \vee r)$ and $Z \geqslant i^r(q \vee r)$ by (4). By (3), $Y \sim Z$.

(iii) Denote the equivalence class of X by $[X]$, and let $\mathbf{H}$ denote the set of equivalence classes, together with a new symbol 0. Define $[X] > 0$, and $[X] \geqslant [X']$ whenever $X, X' \in \mathbf{E}(\mathbf{G}^p)$ for some p and $X \geqslant X'$. For this to be meaningful, we need to know that $[X] = [Y]$, $[X'] = [Y']$, and $Y, Y' \in \mathbf{E}(\mathbf{G}^q)$ imply that $Y \geqslant Y'$. But

$$Y = \varphi_{qp}\varphi_{pq}^{-1}(X) \geqslant \varphi_{qp}\varphi_{pq}^{-1}(X') = Y',$$

as required.

Next, suppose $[X] \geqslant [X'] = [Y']$, with $X, X' \in \mathbf{E}(\mathbf{G}^p)$ and $Y' \in \mathbf{E}(\mathbf{G}^q)$. We claim that there is a $Y \in \mathbf{E}(\mathbf{G}^q)$ for which $[X] = [Y]$. For, $[X'] = [Y']$ implies that $X \geqslant X' \geqslant i^p(p \vee q)$, so $X \in \operatorname{Im} \varphi_{pq}$. Consequently, $Y = \varphi_{qp}\varphi_{pq}^{-1}X$ has the desired properties.

It follows that $\mathbf{H}$ is a poset.

Write $p^* = [i^p(p)] = \{i^p(p)\} \in \mathbf{H}$, and consider $\mathbf{H}^{p^*}$. We claim that $X \to [X]$ is an isomorphism of $\mathbf{E}(\mathbf{G}^p)$ onto $\mathbf{H}^{p^*}$. For, $X \geqslant i^p(p)$ (the 0-element of $\mathbf{E}(\mathbf{G}^p)$) implies that $[X] \geqslant p^*$. Order is clearly preserved. $[X] = [Y]$ with $X, Y \in \mathbf{E}(\mathbf{G}^p)$ implies that $X = \varphi_{pp}^{-1}(X) = \varphi_{pp}^{-1}(Y) = Y$ (as $\varphi_{pp} = 1$). Moreover, $[X] \in \mathbf{H}^{p^*}$ means that $[X] \geqslant p^* = [i^p(p)]$, so $[X] = [X']$ for some $X' \geqslant i^p(p)$. This proves our claim.

For $W \in \mathbf{G}^p$, define $\theta^p(W) = [i^p(W)]$, so $\theta^p\colon \mathbf{G}^p \to \mathbf{H}^{p^*}$ preserves order. Moreover, θ^p is an isometry since i^p is.

There is a natural extension $\theta\colon \mathbf{G} \to \mathbf{H}$ of the maps θ^p defined by

$\theta(0) = 0$ and $\theta(W) = \theta^p(W)$ if $W \geqslant p$. This map is well-defined: $W \geqslant p, q$ implies $i^p(W) \geqslant i^p(p \vee q)$, so by (1) and (2), $\varphi_{pq}^{-1} i^p(W) = i^{p \vee q}(W) = \varphi_{qp}^{-1} i^q(W)$, and hence $\theta^p(W) = [i^p(W)] = [i^q(W)] = \theta^q(W)$ by (3).

θ is an order-monomorphism. For, $W \geqslant 0$ implies $\theta(W) \geqslant \theta(0)$, while $W \geqslant U \geqslant p$ implies $i^p(W) \geqslant i^p(U)$, and hence $\theta^p(W) = [i^p(W)] \geqslant [i^p(U)] = \theta^p(U)$. Since each θ^p is injective, so is θ.

(iv) We have now embedded **G** in **H**. We next show that **H** *is a lattice*. First, let us show that $p^* \vee q^*$ exists and

$$p^* \vee q^* = [i^p(p \vee q)] = [i^q(p \vee q)]. \tag{6}$$

For, $[i^p(p \vee q)] \geqslant [i^p(p)] = p^*$, while (by (1) and (2)) $\varphi_{pq}^{-1} i^p(p \vee q) = i^{p \vee q}(p \vee q) = \varphi_{qp}^{-1} i^q(p \vee q)$, so $[i^p(p \vee q)] = [i^q(p \vee q)] \geqslant q^*$. Suppose $[X] \geqslant p^*, q^*$. Then we may assume that $[X] = [X']$ with $X \in \mathbf{E}(\mathbf{G}^p)$ and $X' \in \mathbf{E}(\mathbf{G}^q)$. By (3), $X \geqslant i^p(p \vee q)$, so $[X] \geqslant [i^p(p \vee q)]$. This proves (6).

Now consider any $[X], [Y] \in \mathbf{H} - \{0\}$. If $[X], [Y] \geqslant p^*$, then $[X] \vee [Y]$ and $[X] \wedge [Y]$ exist by the structure of $\mathbf{H}^{p^*}$. Consider the remaining case. Here it is clear that $[X] \wedge [Y] = 0$. Let $X \in \mathbf{E}(\mathbf{G}^p)$, $Y \in \mathbf{E}(\mathbf{G}^q)$, and write $Y' = Y \vee i^q(p \vee q) \in \mathbf{E}(\mathbf{G}^q)$. Then $[Y'] \geqslant [i^q(p \vee q)] \geqslant p^*$ by (6), so $[X] \vee [Y']$ exists and is $\geqslant [X], [Y]$. Suppose $[Z] \geqslant [X], [Y]$. Then $[Z] \geqslant p^*, q^*$, so (by (6)) $[Z] \geqslant [Y] \vee [i^q(p \vee q)] = [Y']$, and hence $[Z] \geqslant [X] \vee [Y']$. This proves that **H** is a lattice. We remark that it has a 1, namely $[1^p]$, where 1^p is the appropriate element of $\mathbf{E}(\mathbf{G}^p)$.

Note that $\theta(\mathbf{G})$ is the set of elements of **H** which are joins of points. Note also that **H** has a natural dimension function, defined by $\dim 0 = -1$ and $\dim[X] = \dim X + 1$ when $X \in \mathbf{E}(\mathbf{G}^p)$. (This is well-defined: if $[X] = [Y]$, $Y \in \mathbf{E}(\mathbf{G}^q)$, then $\dim X = \dim \varphi_{pq}^{-1} X + 1 = \dim \varphi_{qp}^{-1} Y + 1 = \dim Y$.) Moreover, θ is an isometry.

(v) We next prove the following universal property of **H**. Let $\alpha\colon \mathbf{G} \to \mathbf{M}$ be an isometry from **G** into a projective K-space **M**. Then there is a unique isometry $\beta\colon \mathbf{H} \to \mathbf{M}$ such that

$$\begin{tikzcd} \mathbf{G} \arrow[rr, "\theta"] \arrow[dr, "\alpha"'] & & \mathbf{H} \arrow[dl, "\beta"] \\ & \mathbf{M} & \end{tikzcd} \tag{7}$$

is commutative.

To prove the existence (and uniqueness) of β in (7), for each p let α^p

denote the restriction of α to $\mathbf{G}^p$. There is a unique isometry $\beta^p\colon \mathbf{G}^p \to \mathbf{M}^{\alpha(p)}$ such that

$$\begin{tikzcd} \mathbf{G}^p \arrow[rr, "i^p"] \arrow[dr, "\alpha^p"'] & & \mathbf{E}(\mathbf{G}^p) \arrow[dl, "\beta^p"] \\ & \mathbf{M}^{\alpha(p)} & \end{tikzcd} \tag{8}$$

is commutative. Let β^{pq} denote the restriction of β^p to $\mathbf{E}(\mathbf{G}^p)^{i^p(p\vee q)}$.

We require the relation

$$\beta^{pq}\varphi_{pq} = \beta^{qp}\varphi_{qp}\,. \tag{9}$$

Let $\alpha^{p\vee q}$ denote the restriction of α to $\mathbf{G}^{p\vee q}$. Then we have the following diagram:

$$\begin{tikzcd} \mathbf{G}^{p\vee q} \arrow[rr, "i^{p\vee q}"] \arrow[dr, "\alpha^{p\vee q}"'] & & \mathbf{E}(\mathbf{G}^{p\vee q}) \\ & \mathbf{M}^{\alpha(p\vee q)} & \end{tikzcd}$$

By (1) and (8), if $W \in \mathbf{G}^{p\vee q}$, then

$$\beta^{pq}\varphi_{pq}i^{p\vee q}(W) = \beta^{pq}i^p(W) = \alpha^p(W) = \alpha^{p\vee q}(W).$$

Thus, $\beta^{pq}\varphi_{pq}i^{p\vee q} = \alpha^{p\vee q} = \beta^{qp}\varphi_{qp}i^{p\vee q}$. Since the above diagram can be completed in just one way (by the definition of envelopes), (9) follows.

We now define β by $\beta(0) = 0$ and $\beta([X]) = \beta^p(X)$ if $X \in \mathbf{E}(\mathbf{G}^p)$. Suppose $[X] = [Y]$ with $X \in \mathbf{E}(\mathbf{G}^p)$ and $Y \in \mathbf{E}(\mathbf{G}^q)$. Then $\varphi_{pq}^{-1}(X) = \varphi_{qp}^{-1}(Y)$, $X \in \mathbf{E}(\mathbf{G}^p)^{i^p(p\vee q)}$, and $Y \in \mathbf{E}(\mathbf{G}^q)^{i^q(p\vee q)}$, so $\beta^p(X) = \beta^{pq}\varphi_{pq}\varphi_{qp}^{-1}(Y) = \beta^q(Y)$ by (9). Consequently, β is well-defined.

Since each β^p is an isometry, so is β. This completes the existence half of (7). For uniqueness, note that (7) will yield (8) by restriction. Thus, since β^p was uniquely determined in (8), β is unique in (7).

(vi) The proof of Theorem 1 will be completed by applying the Embedding Lemma of [3] to $\mathbf{H}$. We must check the following properties of $\mathbf{H}$. $\mathbf{H}$ is a poset. Each $[X] \in \mathbf{H}$ has a dimension. We can thus speak of points, lines, planes, and 3-spaces of $\mathbf{H}$. The required axioms are as follows.

(E1) For each point p^*, the poset $\mathbf{H}^{p^*}$ is a projective K-space of dimension $\geqslant 3$.

(E2) Two distinct points are on a unique line; no point is on any other point.

(E3) If two distinct planes are on (at least) two points, they are on a 3-space.

(E4) Each line plane, and 3-space is on at least one point.

(E5) No element of dimension $\leqslant 3$ is on all points.

All these properties are obvious. For example, consider (E3). Let $[E_1]$ and $[E_2]$ be planes on p^* with $[E_1] \wedge [E_2]$ a line. By (*iv*), $[E_1] \vee [E_2]$ exists, and by considering $\mathbf{H}^{p^*}$ we find it is a 3-space.

Now [3] yields a modular geometric lattice $\mathbf{N}$, and a monomorphism φ from the points, lines, planes, and 3-spaces of $\mathbf{H}$ into $\mathbf{N}$, such that (I) φ maps points, lines, planes, and 3-spaces to elements of the same dimension while preserving order, (II) for each point p^* of $\mathbf{H}$, $\varphi(\mathbf{H}^{p^*})$ contains all lines and planes of $\mathbf{N}$ on $\varphi(p^*)$, and (III) $\dim \mathbf{N} = \dim \mathbf{H}^{p^*} + 1$. Clearly, $\mathbf{N}$ is a finite-dimensional projective K-space, and φ extends to a unique order-preserving monomorphism—also called φ—from $\mathbf{H}$ to $\mathbf{N}$, which preserves dimension.

Consequently, φ induces an isomorphism from $\mathbf{H}^{p^*}$ onto $\mathbf{N}^{\varphi(p^*)}$. If $0 \neq [X], [Y] \in \mathbf{H}$, and $p^* \leqslant [X]$, then $\varphi([X] \vee [Y]) = \varphi([X] \vee [Y] \vee p^*) = \varphi([X]) \vee \varphi([Y] \vee p^*)$; but, if $q^* \leqslant [Y]$, then

$$\begin{aligned}\varphi([Y] \vee p^*) &= \varphi([Y] \vee p^* \vee q^*) = \varphi([Y]) \vee \varphi(p^* \vee q^*)\\ &= \varphi([Y] \vee \varphi(p^*) \vee \varphi(q^*)).\end{aligned}$$

Thus, φ is a join-monomorphism, and hence an isometry since $\varphi(1) = 1$.

(vii) It remains only to show that, corresponding to any isometry $\beta\colon \mathbf{H} \to \mathbf{M}$, with $\mathbf{M}$ a projective K-space, there is a unique isometry $\gamma\colon \mathbf{N} \to \mathbf{M}$ making

$$\begin{tikzcd} \mathbf{H} \arrow[rr, "\varphi"] \arrow[dr, "\beta"'] & & \mathbf{N} \arrow[dl, "\gamma"] \\ & \mathbf{M} & \end{tikzcd} \tag{10}$$

commutative.

For each point x of $\mathbf{N}$, let $\varphi'(x)$ be the set of lines $[X]$ of $\mathbf{H}$ with $\varphi([X]) > x$. Then $\varphi'(x)$ is a set of pairwise coplanar lines of $\mathbf{H}$ such that (by property (II) of $\mathbf{N}$) each point of $\mathbf{H}$ is on one of these lines. Similarly, if L is a line of $\mathbf{N}$, let $\varphi'(L)$ be the set of planes $[X]$ of $\mathbf{H}$ with $\varphi([X]) > L$. Then $\varphi'(L)$ is a set of planes, any two of which span a 3-space of $\mathbf{H}$, such that each point of $\mathbf{H}$ is on one of these planes.

Now define $\gamma(x) = \wedge\beta\varphi'(x)$ and $\gamma(L) = \wedge\beta\varphi'(L)$ for each point x

and line L of $\mathbf{N}$. We claim that $\gamma(x)$ is a point and $\gamma(L)$ is a line. For example, consider $\beta\varphi'(L)$. This is a set of planes of $\mathbf{M}$ such that any two span a 3-space and each point of $\beta(H)$ is on one of these planes. In $\mathbf{M}$, any two of these planes meet in a line. Suppose three of these planes E_1, E_2, E_3 of $\mathbf{M}$ satisfy $E_3 \not> E_1 \wedge E_2$. Clearly, $E_1 \wedge E_2 \wedge E_3$ is a point of $\mathbf{M}$ and $\dim(E_1 \vee E_2 \vee E_3) = 3 < \dim \beta(\mathbf{H})$. We can thus find $E \in \beta\varphi'(L)$ with $E \not< E_1 \vee E_2 \vee E_3$. However, once again $\dim(E \vee E_i \vee E_j) = 3$ whenever $1 \leqslant i < j \leqslant 3$, so each $E \wedge (E_i \vee E_j)$ is a line. Since these lines span E, we must have $E < E_1 \vee E_2 \vee E_3$, which is not the case. This proves that $\gamma(L)$ is a line. The proof that $\gamma(x)$ is a point is similar.

Suppose $x < L$. Then each plane in $\varphi'(L)$ is on a line in $\varphi'(x)$. (This requires property (II) of $\mathbf{N}$.) Since the same must hold for $\beta\varphi'(L)$ and $\beta\varphi'(x)$, $\gamma(x) < \gamma(L)$.

It is easy to see $x \to \varphi'(x)$ and $L \to \varphi'(L)$ are injective. So are $x \to \gamma(x)$ and $L \to \gamma(L)$. For example, consider $\gamma(L)$. Any two planes of $\varphi\varphi'(L)$ meet in L, and any two planes of $\beta\varphi'(L)$ meet in $\gamma(L)$. Thus, distinct lines L determine sets $\varphi'(L)$ having at most one common plane, and hence determine different lines $\gamma(L)$.

There is now an obvious and unique extension of γ to an isometry $\gamma\colon \mathbf{N} \to \mathbf{M}$. We claim that (10) holds. For, if p^* is any point of H, then $\varphi'\varphi(p^*)$ consists of all the lines in $\mathbf{H}^{p^*}$, and hence $\beta\varphi'\varphi(p^*)$ consists of lines on $\beta(p^*)$. Consequently, $\gamma\varphi(p^*) = \beta(p^*)$. Also, if $[Z]$ is any line of $\mathbf{H}$, then $\varphi'\varphi([Z])$ consists of all planes of $\mathbf{H}$ on $[Z]$, so $\beta\varphi'\varphi([Z])$ consists of planes on $\beta([Z])$. Consequently, $\gamma\varphi([Z]) = \beta([Z])$. Thus, $\gamma\varphi$ and β are isometries $\mathbf{H} \to \mathbf{M}$ which agree on points and lines. Restricting to each $\mathbf{H}^{p^*}$, we find that $\gamma\varphi = \beta$.

Finally, γ is unique in (10). For, suppose (10) holds. Let x be any point of $\mathbf{N}$, and let $[X] \in \varphi'(x)$, so $[X] \in \mathbf{H}$ is a line and $\varphi([X]) > x$. Then $\gamma(x) < \gamma\varphi([X]) = \beta[X]$ for all $[X] \in \varphi'(x)$. Consequently, $\gamma(x) = \wedge\beta\varphi'(x)$ as before. Similarly, $\gamma(L)$ must be $\wedge\beta\varphi'(L)$. This completes the proof of the Main Lemma.

4. Main Results

Proof of Theorem 1. We use induction on $\dim \mathbf{G}$. If $\dim \mathbf{G} = j + 2$, there is nothing to prove. Assume $\dim \mathbf{G} > j + 2$. By induction, $\mathbf{G}^p$ has a K-envelope as $\dim \mathbf{G}^p \geqslant j + 2$. The same will be true of $\mathbf{G}^{p\vee q}$, except perhaps if $\dim \mathbf{G}^{p\vee q} = j + 1$, in which case hypothesis (a) applies. Similarly, $\mathbf{G}^{p\vee q\vee r}$ will have a K-envelope, except perhaps if $\dim \mathbf{G}^{p\vee q\vee r} = j + 1$ or j, in which case (a) or (b) applies. In any case, by Lemma 6, $\mathbf{G}^{p\vee q\vee r}$ is K-rigid. Consequently, by the Main Lemma, $\mathbf{G}$ has a K-envelope.

THEOREM 3. *Suppose K has no proper subfield isomorphic to itself. Let $j \geqslant 1$, and let $\mathbf{G}$ be a geometric lattice with* $\dim \mathbf{G} \geqslant j + 2$. *Suppose $j > 1$ if* $\operatorname{Aut}(K) \neq 1$. *Assume that, for all $W \in \mathbf{G}$,*

(a′) *$\mathbf{G}^W$ has a K-envelope whenever* $\dim \mathbf{G}^W = j$ *or* $j + 1$, *and*

(b′) *there is an isometry from $\mathbf{G}^W$ into a projective K-space whenever* $\dim \mathbf{G}^W = j + 2$.

Then $\mathbf{G}$ has a K-envelope.

Proof. By Lemmas 6 and 7, (b) of Theorem 1 holds. Apply Theorem 2 to $\mathbf{G}^W$ whenever $\dim \mathbf{G}^W = j + 2$ to see that (a) also holds.

5. APPLICATIONS

As a first example, consider a finite geometric lattice $\mathbf{G}$ of dimension $\geqslant 4$ such that $\mathbf{G}^W$ is an inversive plane whenever $\dim \mathbf{G}^W = 3$. All these inversive planes then have the same order n. It is easy to check that such an inversive plane is egglike [2, p. 254] if and only if n is a prime power and $\mathbf{G}$ has a $GF(n)$-envelope $(PG(3, n), i)$. This is the case when $n \leqslant 3$ ([2, p. 273]); by a basic result of Dembowski [1], it is also the case whenever n is even. Also, in these cases it is clear that each affine plane $\mathbf{G}^W$ (where $\dim \mathbf{G}^W = 2$) has a $GF(n)$-envelope. Finally, when $\dim \mathbf{G}^W = 1$, $\mathbf{G}^W$ has $n + 1$ points, and hence is certainly $GF(n)$-rigid. Consequently, Theorem 1 applies for $n = 3$ or n even. When $n = 3$ and $\mathbf{G}$ is the 5-dimensional lattice associated with the Mathieu group M_{12}, we find that $\mathbf{G}$ has a $GF(3)$-envelope $(PG(5, 3), i)$. Consequently, by Lemma 5, M_{12} is contained in $PGL(6, 3)$, as is well-known. Similarly, when n is even we can obtain the result proved in [3, Section 5, Remark 2].

The following is quite a different application.

THEOREM 4. (Tutte). *Let $\mathbf{G}$ be a finite geometric lattice each of whose colines is on at most 3 hyperplanes. Then $\mathbf{G}$ has a $GF(2)$-envelope.*

Proof. This is clear for $\dim \mathbf{G} \leqslant 1$, and easy to check for $\dim \mathbf{G} = 2$. It is straightforward to check that, when $\dim \mathbf{G} = 3$, there is an isometry from $\mathbf{G}$ into $PG(3, 2)$. The result now follows from Theorem 3.

Our results are not strong enough to prove either Tutte's representation theorem for unimodular lattices or an analogue of Theorem 4 for representations over $GF(3)$ (see [4]). The reason is that, in either case, the lattice does not always have a K-envelope.

It is probably worth mentioning that, in the definition of K-envelopes in Section I, it would have been pointless to require that β be induced by a linear transformation (as opposed to a semi-linear one). For, assume $\mathrm{Aut}(K) \neq 1$ and $\mathbf{G} = PG(n, K)$, $n \geqslant 2$. If i is the identity map on G, then $(\mathbf{G}, i)$ should be the appropriate envelope. Yet, if in the definition $\alpha: \mathbf{G} \to \mathbf{G}$ is not linear, then neither is β (i.e., α^{-1}).

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