JOURNAL OF COMBINATORIAL THEORY, Series A 38, 66–74 (1985)

Homogeneous Designs and Geometric Lattices

WILLIAM M. KANTOR*

Department of Mathematics, University of Oregon, Eugene, Oregon 97403

Communicated by F. Buekenhout

Received January 28, 1983

1. INTRODUCTION

During the last 20 years, there has been a great deal of research concerning designs with $\lambda = 1$ admitting 2-transitive groups. The following theorems will be proved in this note; they are fairly simple consequences of the classification[1] of all finite simple groups (see, e.g., [6]).

THEOREM 1. *Let $\mathscr{D}$ be a design with $\lambda = 1$ admitting an automorphism group 2-transitive on points. Then $\mathscr{D}$ is one of the following designs:*

(i) $PG(d, q)$,

(ii) $AG(d, q)$,

(iii) *The design with* $v = q^3 + 1$ *and* $k = q + 1$ *associated with* $PSU(3, q)$ *or* $^2G_2(q)$,

(iv) *One of two affine planes, having* 3^4 *or* 3^6 *points* [5, p. 236], *or*

(v) *One of two designs having* $v = 3^6$ *and* $k = 3^2$ [12].

THEOREM 2. *Let $\mathscr{L}$ be a finite geometric lattice of rank at least* 3 *such that* Aut $\mathscr{L}$ *is transitive on ordered bases. Then either*

(i) $\mathscr{L}$ *is a truncation of a Boolean lattice or a projective or affine geometry,*

(ii) $\mathscr{L}$ *is the lattice associated with a Steiner system* $S(3, 6, 22)$, $S(4, 7, 23)$, *or* $S(5, 8, 24)$, *or*

(iii) $\mathscr{L}$ *is the lattice associated with the* 65*-point design for* $PSU(3, 4)$.

The groups in Theorems 1 and 2 are described in the course of the proof. It would, of course, be desirable to have more elementary proofs of both

* This research was supported in part by NSF Grant MCS 7903130-82.

[1] At the time of writing (December 1982), this classification is not quite complete: the uniqueness of the Monster has not been proved. However, this does not cause any difficulties with our use of the classification.

0097-3165/85 $3.00

theorems. Unfortunately, even the determination of all 2-transitive collineation groups of $AG(d, q)$ seems to require the classification of all finite simple groups. (The case $PG(d, q)$ is much simpler [2].)

The above theorems were proved more than three years ago, on the assumption that the aforementioned classification would be completed. Since then, special cases have appeared: Buekenhout [1, Sect. 4], Key and Shult [14], Hall [7], and Cherlin, Harrington, and Lachlan [3]. None of these is used in our proof; each also assumes the aforementioned classification. I am grateful to F. Buekenhout and P. Seymour for urging that I write up the proofs of Theorems 1 and 2.

2. Preliminaries

Let G be a 2-transitive group of permutations of a set X of size v. If $Y \subseteq X$ let G_Y be its set stabilizer, $G(Y)$ its pointwise stabilizer, and set $G_Y^Y = G_Y/G(Y)$.

If $S \subseteq G$ let $F(S)$ be the set of fixed points of S.

All other notation is very standard.

The classification of all finite 2-transitive groups is a consequence of the classification of all finite simple groups: see [4, 9–11, 13, 15]. (Note, however, that not all sporadic simple groups were dealt with in those references. These are not difficult to eliminate by using properties of the individual groups [6] and imitating those references, especially [10, 11].) The list of groups is as follows.

(A) *G has a simple normal subgroup N, and $N \leqslant G \leqslant \operatorname{Aut} N$, where N and v are as follows:*

(1) A_v, $v \geqslant 5$.

(2) $PSL(d, q)$, $d \geqslant 2$, $v = (q^d - 1)/(q-1)$ (*two representations if* $d > 2$); *here*, $(d, q) \neq (2, 2), (2, 3)$.

(3) $PSU(3, q)$, $v = q^3 + 1$, $q > 2$.

(4) $Sz(q)$, $v = q^2 + 1$, $q = 2^{2e+1} > 2$.

(5) ${}^2G_2(q)'$, $v = q^3 + 1$, $q = 3^{2e+1}$.

(6) $Sp(2n, 2)$, $n \geqslant 3$, $v = 2^{2n-1} \pm 2^{n-1}$.

(7) $PSL(2, 11)$, $v = 11$ (*two representations*).

(8) *Mathieu groups* M_v, $v = 11, 12, 22, 23, 24$ (*two representations for* M_{12}).

(9) M_{11}, $v = 12$.

(10) A_7, $v = 15$ (*two representations*).

(11) *HS* (*Higman–Sims group*), $v = 176$ (*two representations*).

(12) .3 (*Conway's smallest group*), $v = 276$.

(B) *G has a regular normal subgroup N which is elementary abelian of order* $v = p^d$, *where p is a prime. Identify G with a group of affine transformations* $x \to x^g + c$ *of* $GF(p)^d$, *where* $g \in G_0$. *Then one of the following occurs*:

(1) $G \leqslant A\Gamma L(1, v)$.

(2) $G_0 \trianglerighteq SL(n, q)$, $q^n = p^d$.

(3) $G_0 \trianglerighteq Sp(n, q)$, $q^n = p^d$.

(4) $G_0 \trianglerighteq G_2(q)'$, $q^6 = p^d$, *q even*,

(5) $G_0 \cong A_6$ *or* A_7, $v = 2^4$.

(6) $G_0 \trianglerighteq SL(2, 3)$ *or* $SL(2, 5)$, $v = p^2$, $p = 5, 7, 11, 19, 23, 29$, *or* 59 *or* $v = 3^4$.

(7) G_0 *has a normal extraspecial subgroup E of order* 2^5, *and* G_0/E *is isomorphic to a subgroup of* S_5, *where* $v = 3^4$.

(8) $G_0 = SL(2, 13)$, $v = 3^6$.

The remark "two representations" refers to the fact that there are two different 2-transitive permutation representations of degree v, and these are interchanged by an outer automorphism of G.

Almost all of the examples on the above lists are familiar in various contexts. We will need only a few properties of each one, especially the orbit-lengths of the stabilizer G_{xy} of two different points x and y. In almost every case, the reader should have no trouble bounding these lengths as required in the next two sections. The lengths are relevant because of Lemma 2.1 below.

Throughout Sections 3 and 4, G will be 2-transitive on the set X of points of the design $\mathscr{D}$, where $\lambda = 1$ and $k > 2$. Let x and y be as above, and let B be the block on x and y. Then G_{xy} fixes B, and hence acts on $B - \{x, y\}$. Consequently, G_{xy} must have a fairly short orbit on $X - \{x, y\}$, in view of the following standard, elementary facts.

LEMMA 2.1. (i) *There are* $r = (v-1)/(k-1)$ *blocks per point.*

(ii) *Either* $v = k^2 - k + 1$ *or* $v \geqslant k^2$.

3. SIMPLE NORMAL SUBGROUP

In this section we will begin the proof of Theorem 1, assuming that G has a simple normal subgroup N. We will run through the list of possibilities given in Section 2. In each case, except $G = {}^2G_2(3) \cong P\Gamma L(2, 8)$, N is also 2-transitive on X and we may assume that $G = N$.

Case $G = A_v$. This cannot occur since G is 3-transitive.

Case $G = PSL(2, q), q \geqslant 5$. Since all orbits of G_{xy} on $X - \{x, y\}$ have size $\geqslant (q-1)/2$, Lemma 2.1 yields a contradiction.

Case $G = PSL(d, q), d \geqslant 3$. Here G_{xy} has orbit-lengths $q-1$ and $v-(q+1)$ on $X - \{x, y\}$. By (2.1), $|B| = q+1$. Thus, $\mathscr{D} = PG(d-1, q)$.

Case $G = PSU(3, q), v = q^3 + 1$. Each orbit of G_{xy} on $X - \{x, y\}$ has length $q-1$ or at least $(q^2-1)/3$. By Lemma 2.1, $k = 2 + (q-1)$. Then $\mathscr{D}$ is the usual design for G.

Case $G = Sz(q), v = q^2 + 1$. Each orbit of G_{xy} on $X - \{x, y\}$ has length $q-1$, so that Lemma 2.1 yields a contradiction.

Case $G = {}^2G_2(q)$. Here $|G_{xB}| = q^3(q-1)/r$ and $q^3 = v - 1 = r(k-1)$. It follows that $k-1$ is a power of 3, and that G_{xB} has a normal 3-subgroup transitive on $B - \{x\}$. A Sylow 2-subgroup of G_B^B is elementary abelian of order $\leqslant 8$. By Section 2, $G_B^B \geqslant PSL(2, k-1)$ or ${}^2G_2(k-1)$.

There is a unique involution t in G_{xy}, $F = F(t)$ has size $q+1$, and $C(t)^F = PSL(2, q)$. If $B = F$ then (iii) holds. If $B \subset F$ then F is a subdesign, and $C(t)^F$ yields a contradiction.

Assume that $B \not\subset F$. Every element in $G_{xy} - \langle t \rangle$ has fixed point set $\{x, y\}$. Thus, G_{xy} is faithful on B. Also, $|G_{xy}| = q-1$. This rules out ${}^2G_2(k-1)$, and shows that $k = q$ and $G_B^B = PGL(2, q)$. But then G_B^B contains a dihedral group of order 8. This contradiction shows that (iii) is the only possibility in this case.

Case $G = Sp(2n, 2)$ *and* $v = 2^{n-1}(2^n \pm 1)$, $n \geqslant 3$. Here G_x acts on $X - \{x\}$ as $O^{\pm}(2n, 2)$ does on its singular vectors. Then G_{xy} has orbit-lengths $2(2^{n-1} \mp 1)(2^{n-2} \pm 1)$ and 2^{2n-2} on $X - \{x, y\}$, which is impossible by Lemma 2.1.

Case $G = A_7, v = 15$. Since G_{xy} has orbit-lengths 1 and 12 on $X - \{x, y\}$, $\mathscr{D} = PG(3, 2)$.

Case $G = PSL(2, 11), v = 11$. Since G_{xy} has orbit-lengths 3 and 6 on $X - \{x, y\}$, this case cannot occur by Lemma 2.1.

Case $G = M_{11}, M_{12}, M_{22}, M_{23}$, or M_{24}. Since G is 3-transitive, these cannot occur.

Case $G = HS, v = 176$. Since G_{xy} has orbit-lengths 12, 72, and 90 on $X - \{x, y\}$, $k = 2 + 12$ by Lemma 2.1. But then $r = (v-1)/(k-1)$ is not an integer.

Case $G = .3, v = 276$. Since G_{xy} has orbit-lengths 112, 162 on $X - \{x, y\}$, Lemma 2.1 again produces a contradiction.

4. Regular Normal Subgroup

Next, assume that G has a regular normal subgroup N of order p^d. As in Section 3, we can replace G by a 2-transitive subgroup if necessary.

The only interesting part of the proof of Theorem 1 is the following case.

Proposition 4.1. *If $G \leqslant A\Gamma L(1, v)$ then $\mathscr{D}$ is an affine space.*

We may identify X with $GF(v)$. Let B be the block containing 0 and 1. It suffices to show that B is a subfield of X.

Set $G^* = G \cap AGL(1, v)$.

Lemma 4.2. *B is a subspace of X.*

Proof. Since G_B^B is 2-transitive, $(G_B^B)'$ is transitive. Also, $(G_B)' < G^*$. If $p|k$, it follows that B is a subspace.

If $p \nmid k$, then a regular normal subgroup of G_B^B is cyclic. Then k is a prime and $G_{01}^B = 1$. Set $F = F(G_{01})$. Then G_F^F is 2-transitive [16, (9.4)] and $B \subseteq F$. Since F is a subspace of X, it follows that F is a subdesign. By induction, $F = X$. Then G is sharply 2-transitive. Since G_B has a dihedral subgroup of order $2k$, this is impossible [17, p. 196; 5, (5.2.4)].

Lemma 4.3. *We may assume that G_B is faithful on B.*

Proof. Assume that $G(B) \neq 1$. Then $F = F(G(B))$ is a subfield of X, and G_F^F is 2-transitive [16, (9.4)]. If $B = F$ we are finished. If $B \subset F$ then, assuming inductively that Proposition 4.1 holds for smaller v, we see that B is a subfield of F and hence of X.

Lemma 4.4. *If $(G^*)_{0B}$ is irreducible on B, then $\mathscr{D}$ is an affine space.*

Proof. Let K be the $GF(p)$-space of linear transformations spanned by $(G^*)_{0B}$. Then K is a subfield of V, and K fixes B. By hypothesis, $K = GF(k)$. Thus, $B = K$.

Lemma 4.5. *G_B is isomorphic to a subgroup of $A\Gamma L(1, k)$.*

Proof. G_B^B is a 2-transitive group such that G_{0B}^B is metacyclic. Any such group of degree k lies in $A\Gamma L(1, k)$.

Remarks. The preceding lemma does not assert that the $A\Gamma L(1, k)$ is embedded in $A\Gamma L(1, v)$ in the natural manner. The remainder of the proof of Proposition 4.1 is, in fact, concerned with proving just such an embedding.

Conceivably, $G_{01} = 1$. When this happens, the following all hold [17, p. 190; 5, p. 229]:

LEMMA 4.6. (i) *If* $|Z(G_0)| = q-1$ *and* $|G : G^*| = n$ *then* $v = q^n$;

(ii) $Z(G_0) \leqslant (G^*)_0$;

(iii) *Every prime divisor of n also divides* $q-1$; *and*

(iv) *If* $q \equiv 3 \pmod 4$ *then* $n \not\equiv 0 \pmod 4$.

Proof of Proposition 4.1. Write $k = p^e$. First assume that there is a prime s such that $s \mid p^e - 1$ but $s \nmid p^i - 1$ for $1 \leqslant i < e$. Then $e \mid s-1$, so that $s \nmid e$. By Lemma 4.5, a Sylow s-subgroup S of G_{0B} lies in $AGL(1, k)$. Each nontrivial field automorphism of $GF(k)$ acts nontrivially on S. Thus, if G_B is not $AGL(1, k)$ then $S \leqslant (G_B)' \leqslant G' \leqslant G^*$ and Lemma 4.4 applies. Consequently, assume that G_B is $AGL(1, k)$. Then $G_{01} = 1$ and Lemma 4.6 can be used. If $s \nmid n$ then $S \leqslant G^*$ by Lemma 4.6(i). If $s \mid n$ then $s \mid q-1$ by Lemma 4.6(iii), and hence $S \leqslant Z(G_0)$ since Sylow s-subgroups of G_0 are cyclic. Consequently, $S \leqslant (G^*)_{0B}$ once again (by Lemma 4.6(ii)).

Now we may assume that no prime s exists. By [18], either $k = p^2$ and p is a Mersenne prime, or else $k = 2^6$.

Let $k = p^2$. If $G_{01} \neq 1$ then $|G_{01}| = 2$ by Lemma 4.5, and $4 \mid |(G_{0B})'|$. Thus $4 \mid |(G_{0B})^*|$, and Lemma 4.4 applies. Now assume that $G_{01} = 1$, and let q and n be as in Lemma 4.6(i). If $q \equiv 3 \pmod 4$ then $p+1 \mid |(G^*)_{0B}|$ by Lemma 4.6(iv), so that Lemma 4.4 applies . Suppose that $q \equiv 1 \pmod 4$. Since $p \equiv 3 \pmod 4$ it follows that $p^2 - 1 \mid q-1$. Then $p^2 - 1 \mid |Z(G_0)|$. In particular, a Sylow 2-subgroup of G_0 has a center of order $\geqslant 2(p+1)$, and hence cannot be generalized quaternion and so must be cyclic. Since $|G_{0B}| = p^2 - 1$, a Sylow 2-subgroup of G_{0B} must lie in $Z(G_0)$, and hence also in G^* (by Lemma 4.6(ii)). Once again Lemma 4.4 applies.

Finally, consider the case $k = 2^6$. Since $O^2(G)$ is still 2-transitive we may assume that $G = O^2(G)$. Then $|G_{01}| \mid 3$. If $G_{01} \neq 1$ then $|(G_{0B})'| \geqslant 63/3$ and Lemma 4.4 applies. Assume that $G_{01} = 1$. Let $s \in \{3, 7\}$, and let S be a subgroup of G_{0B} of order s. If $s \nmid n$ in Lemma 4.6(i) then $S \leqslant G^*$. If $s \mid n$ then $s \mid q-1$ by Lemma 4.6(iii), and $S \leqslant Z(G_0) \leqslant (G^*)_0$ by Lemma 4.6(i,ii). Thus, $|(G^*)_{0B}| \geqslant 21$, and Lemma 4.4 completes the proof of Proposition 4.1.

We will now run through the remaining cases listed in Section 2.

Remark. If $G_0 \trianglerighteq SL(n, q)$, $Sp(n, q)$, or $G_2(q)'$, we may regard X as a $GF(q)$-space.

Case $G_0 \trianglerighteq SL(n, q)$, $v = q^n$. Here G_{0x} has an orbit of length $q^n - q$. By Lemma 2.1, $B \subseteq \langle x \rangle$. If $B = \langle x \rangle$ then $\mathscr{D} = AG(n, q)$. If $B \subset \langle x \rangle$ then, since $G_{\langle x \rangle}^{\langle x \rangle}$ is 2-transitive, $\langle x \rangle$ is a subdesign of $\mathscr{D}$. By Proposition 4.1, this subdesign is $AG(d, s)$ with $s^d = q$. Then the group of scalar transformations induced on X by $GF(s)$ also acts on each subdesign $\langle x \rangle$ and hence on $\mathscr{D}$. Thus, $\mathscr{D}$ consists of all affine lines over $GF(s)$.

Case $G_0 \trianglerighteq Sp(n, q)$, $n \geqslant 4$, $v = q^n$. This time all orbits of G_{0x} on $V - \langle x \rangle$ have lengths $\geqslant q(q^{n-2} - 1)/(q-1) > q^{n/2}$. By Lemma 2.1, $B \subseteq \langle x \rangle$, and we can proceed as above.

Case $G_0 \trianglerighteq G_2(q)$, *q even*, $v = q^6$. This time all orbits of G_{0x} on $X - \langle x \rangle$ have lengths divisible by $q(q+1)$, $q^3(q+1)$, or q^5. (These are the lengths of the nontrivial orbits of $G_{\langle x \rangle}$ on the 1-spaces of X; see, e.g., [2, (3.1)].) By the above arguments, we may assume that $|B \cap (X - \langle x \rangle)|$ is a nonzero multiple of $q(q+1)$. Note that G_{0B} is transitive on $\Sigma = \{\langle y \rangle \mid y \in B\}$. There is an underlying symplectic structure on X (see, e.g., [2, Appendix]), and $\langle x \rangle$ is the only member of Σ perpendicular to all members of Σ. This contradiction proves that $B \subseteq \langle x \rangle$, and completes this case.

Case $G_0 = G_2(2)'$, $v = 2^6$. Since the orbit lengths of G_{0x} on $X - \langle x \rangle$ are $2(2+1)$, $2^3(2+1)$, 2^4, and 2^4, the preceding argument goes through without any changes.

Case $G_0 \cong A_7$, $v = 2^4$. Since G_{0x} is transitive on $X - \{0, x\}$, this cannot occur.

Case $G_0 \cong A_6$, $v = 2^4$. This time G_{0x} has orbit-lengths 6 and 8 on $X - \{0, x\}$, and Lemma 2.1 yields a contradiction.

Case $v = p^2$, $G_0 \trianglerighteq SL(2, 3)$ *or* $SL(2, 5)$, $p = 5, 7, 11, 19, 23, 29$, *or* 59. A check of the possible groups G shows that we may assume that G has a subgroup H of index $\leqslant 2$ having only one class of involutions. Then H_B contains at least two involutions; since their product is of order p, it follows that $k \geqslant p$. By Lemma 2.1, $k = p$, and then $\mathscr{D}$ is $AG(2, p)$.

Case $v = 3^4$ *and* G_0 *has a normal extraspecial subgroup* E *of order* 2^5. Then $E_x = \langle t \rangle$ with $|t| = 2$ and $|F(t)| = 9$. We have $80 = v - 1 = r(k-1)$, so that $(r, k) = (20, 5)$, $(16, 6)$, $(40, 3)$, or $(10, 9)$.

If $k = 5$ then $|F(t) \cap B| = 3$ and t^B induces a transposition. (We could not have $t^B = 1$ as $F(t)$ would be a subdesign of $\mathscr{D}$.) Then $G_B^B = S_5$. However, G cannot have a subgroup A_5 (although it can have an $SL(2, 5)$). Thus, $G(B)$ contains -1, which is ridiculous.

Similarly, if $k = 6$ then G_B^B is 2-transitive of degree 6, so that $G_B^B \geqslant PSL(2, 5)$. This leads to the same contradiction as above.

If $k = 3$ then $B \subseteq F(t)$. Since $G_{F(t)}^{F(t)}$ is 2-transitive, it follows that B is a 1-space. Then $\mathscr{D} = AG(4, 3)$.

Finally, if $k = 9$ then $\mathscr{D}$ is an affine plane of order 9. By [5, pp. 214, 232, 236], it is the "exceptional nearfied plane."

Case $v = 3^4$ *and* $G_0 \trianglerighteq SL(2, 5)$. The possibilities for r and k are as in the preceding case. As above, $k \neq 5, 6$, while $\mathscr{D}$ is $AG(2, 9)$ or the exceptional nearfield plane if $k = 9$. If $k = 3$ let $t \in G_{0x}$ have order 3. Then $|F(t)| = 9$ and $B \subseteq F(t)$, so that $\mathscr{D} = AG(4, 3)$ as above.

Case $G_0 = SL(2, 13)$, $v = 3^6$. Since G_B contains two involutions, and their product has order 3, we have $1 \neq G_B \cap N \trianglelefteq G_B$. Thus, $G_B \cap N$ is transitive on B, and B is a subspace. If $k = 3$ then $\mathscr{D} = AG(6, 3)$.

If $k = 3^3$ then $|G_{0B}| = 13 \cdot 6$. This uniquely determines G_{0B} (up to conjugacy) and B. The design $\mathscr{D}$ is then the affine plane in [8; 5, p. 236].

Finally, if $k = 3^2$ then $|G_{0B}| = 24$ and [12] applies.

This completes the proof of Theorem 1.

5. *t*-Designs

The list in Section 2 and Theorem 1 easily imply the following:

THEOREM 3. *Let $\mathscr{D}$ be a t-design with $k \geqslant t + 1 \geqslant 4$, and let $G \leqslant \mathrm{Aut}\, \mathscr{D}$ be t-transitive on points. Then either*

(a) *$\mathscr{D}$ consists of the points and planes of $AG(d, 2)$ for some d, and G is $\mathbb{Z}_2^d \rtimes GL(d, 2)$ or $\mathbb{Z}_2^4 \rtimes A_7$ (and $d = 4$);*

(b) *The blocks of $\mathscr{D}$ are all the images of $\{\infty\} \cup GF(q)$ under $PGL(2, q^e)$, $e \geqslant 2$, and $G \trianglerighteq PSL(2, q^e)$; or*

(c) *$\mathscr{D}$ is an $S(4, 5, 11)$, $S(5, 6, 12)$, $S(3, 6, 22)$, $S(4, 7, 23)$, or $S(5, 8, 24)$, and $G \trianglerighteq M_v$.*

Proof. Since G is 3-transitive, the list in Section 2 yields the following possibilities: $\mathbb{Z}_2^d \rtimes GL(d, 2)$, $\mathbb{Z}_2^4 \rtimes A_7$, $PSL(2, v) \leqslant G \leqslant P\Gamma L(2, v)$, or G is a Mathieu group. Assume that $t = 3$. Then Theorem 1 applies to G_x and the corresponding design $\mathscr{D}_x$, and it is straightforward to check that (a), (b), or (c) (with $S(3, 6, 22)$) holds. If $t > 3$ then G is 4-transitive and (c) holds.

6. Proof of Theorem 2

Set $G = \mathrm{Aut}\, \mathscr{L}$, and let k be the common size of all lines of $\mathscr{L}$. Let x be a point of $\mathscr{L}$, and let 1 have the usual meaning for $\mathscr{L}$.

First assume that $k > 2$. The points and lines form a design $\mathscr{D}$ to which Theorem 1 applies. The design for $PSU(3, 2)$ is just $AG(2, 3)$; the design for $PSU(3, 4)$ satisfies the conditions of Theorem 2. Excluding these instances, the examples in (iii)–(v) do not have basis-transitive groups, since $|G_{xy}| < v - k$ in each case. Thus, the points and lines can be identified with the points and lines of a projective or affine geometry. We may assume that *rank*($\mathscr{L}$) > 3, so that (by induction) each interval $[x, 1]$ is a truncation of a projective or affine geometry. Comparison with $\mathscr{D}$ shows that the same is true of $\mathscr{L}$, as required.

Now let $k = 2$, but assume that $\mathscr{L}$ is not the truncation of a Boolean lattice. Then Theorem 3 applies to a suitable truncation of $\mathscr{L}$. On the other hand, by induction $[x, 1]$ is either a truncation of a projective or affine geometry, or a Steiner system as in Theorem 2 (ii). It follows that $\mathscr{L}$ is also either a truncation of a projective or affine geometry or one of the aforementioned Steiner systems.

References

1. F. Buekenhout, Diagram geometries for sporadic groups, *in* "Proceedings, Montreal Monster Conf." to appear.
2. P. J. Cameron and W. M. Kantor, 2-Transitive and antiflag transitive collineation groups of finite projective and polar spaces, *J. Algebra* **60** (1979), 384–422.
3. G. Cherlin, L. Harrington, and A. Lachlan, $\aleph_0$-categorical $\aleph_0$-stable structures, to appear.
4. C. W. Curtis, G. M. Seitz, and W. M. Kantor, The 2-transitive permutation representations of the finite Chevalley groups, *Trans. Amer. Math. Soc.* **218** (1976), 1–59.
5. P. Dembowski, "Finite Geometries," Springer, Berlin/Heidelberg/New York, 1968.
6. D. Gorenstein, "Finite Simple Groups: An Introduction to Their Classification," Plenum, New York, 1982.
7. M. Hall, Jr., Steiner triple systems with a doubly transitive automorphism group, to appear.
8. C. Hering, Eine nicht-desarguesche zweifach transitive affine Ebene der Ordnung 27, *Abh. Math. Sem. Univ. Hamburg* **34** (1969), 203–208.
9. C. Hering, On linear groups which contain an irreducible subgroup of prime order, "Proceedings, Int. Conf. Proj. Planes," pp. 99–105, Washington State Univ. Press, Pullman, 1973.
10. C. Hering, Transitive linear groups and linear groups which contain irreducible subgroups of prime order, *Geom. Dedicata* **2** (1974), 425–460.
11. C. Hering, Transitive linear groups and linear groups which contain irreducible subgroups of prime order, II, to appear.
12. C. Hering, to appear.
13. B. Huppert, Zweifach transitive, auflösbare Permutationsgruppen, *Math. Z.* **68** (1957), 126–150.
14. J. D. Key and E. E. Shult, Steiner triple systems with doubly transitive automorphism groups: A corollary to the classification theorem for finite simple groups, *J. Combin. Theory Ser. A* **36** (1984), 105–110.
15. E. Maillet, Sur les isomorphes holoédriques et transitifs des groupes symétriques ou alternés, *J. Math. Pures Appl.* (5) **1** (1895), 5–34.
16. H. Wielandt, "Finite Permutation Groups," Academic Press, New York/London, 1964.
17. H. Zassenhaus, Über endliche Fastkörper, *Abh. Math. Sem. Univ. Hamburg* **11** (1935), 187–220.
18. K. Zsigmondy, Zur Theorie der Potenzreste, *Monatsh. Math. Phys.* **3** (1892), 265–284.