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k-Homogeneous Groups*

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1. Introduction

A permutation group is called k-homogeneous if it is transitive on the k-sets of permuted points.

Theorem 1. *Let G be a group k-homogeneous but not k-transitive on a finite set Ω of n points, where $n \geqq 2k$. Then, up to permutation isomorphism, one of the following holds:*

(i) $k=2$ *and* $G \leqq A\Gamma L(1, q)$ *with* $n=q\equiv 3 \pmod 4$;

(ii) $k=3$ *and* $PSL(2, q) \leqq G \leqq P\Gamma L(2, q)$, *where* $n-1=q\equiv 3 \pmod 4$;

(iii) $k=3$ *and* $G=AGL(1, 8)$, $A\Gamma L(1, 8)$ *or* $A\Gamma L(1, 32)$; *or*

(iv) $k=4$ *and* $G=PSL(2, 8)$, $P\Gamma L(2, 8)$ *or* $P\Gamma L(2, 32)$.

Here $A\Gamma L(1, q)$ is the group of mappings $x \to a x^{\sigma}+b$ on $GF(q)$, where $a \neq 0$ and b are in $GF(q)$ and $\sigma \in \operatorname{Aut} GF(q)$. $AGL(1, q)$ consists of those mappings with $\sigma=1$. All the groups listed in the theorem are assumed to act in their usual permutation representations.

We note that, conversely, each of (i)–(iv) produces examples of k-homogeneous but not k-transitive groups. Thus, in (i) we need only consider maps of the form $x \to a^2 x+b$. Moreover, $PSL(2, q)$, $q\equiv 3 \pmod 4$, and the groups in (iii) and (iv) meet our requirements.

This theorem completes results of Livingstone and Wagner [8], who showed that k must be at most 4. Clearly $k>1$. For the case $k=2$, see [7], Proposition 3.1. The case $k=4$ was considered in [6], but there is an error in the proof. Note that the hypothesis $n \geqq 2k$ is essential, since, for example, a 2-transitive group of degree n is $(n-2)$-homogeneous.

The case $k=4$ will follow easily from the case $k=3$. If $k=3$ and neither (ii) nor (iii) holds, it is easy to show that $3 \nmid |G|$ and the stabilizer of 2 points has precisely 3 orbits on the remaining points. However, the deep group-theoretic results presently known about $3'$-groups do not seem to apply to our situation. Instead of these we use a combinatorial argument, based on the proof of Gleason's lemma ([4], Lemma 1.7), in order to employ induction.

Our notation is that of Wielandt [9]. If X is a subset of a permutation group then $\Delta(X)$ is its set of fixed points.

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2. Induction

In order to use induction to prove Theorem 1 when $k=3$, we will use somewhat different hypotheses. The result will be an easy consequence of

Theorem 2. *Let G be a group 2-transitive on a finite set Ω, where $n=|\Omega|>2$. Assume:*

(a) $3 \nmid |G|$; *and*

(b) *If $\alpha \neq \beta$ then $G_{\alpha\beta}$ has precisely 3 orbits on $\Omega - \{\alpha, \beta\}$.*

Then $n=5$ and $|G|=20$.

The following result will be used very frequently.

Lemma 1 (Livingstone and Wagner [8], Theorem 3; Bender [1], Lemma 3.3). *Let K be a group transitive on a finite set Φ. Let r be a prime and R an r-subgroup maximal with respect to fixing $\geqq 2$ points. Then $N(R)^{\Delta(R)}$ is transitive.*

Proof of Theorem 2. Let G be a counterexample with n minimal. By (a), $n>5$.

For $\alpha \neq \beta$ let $\Gamma_i(\alpha, \beta)$, $i=1, 2, 3$, be the orbits of $G_{\alpha\beta}$ on $\Omega - \{\alpha, \beta\}$; this labeling is chosen in any way. (These orbits cannot necessarily be labelled so that $\Gamma_i(\alpha^g, \beta^g) = \Gamma_i(\alpha, \beta)^g$ for all $g \in G$, as some g might interchange α and β and also interchange two of the orbits $\Gamma_i(\alpha, \beta)$.) For each i we have $|\Gamma_i(\alpha, \beta)|>1$, as otherwise by (b) $N(G_{\alpha\beta})^{\Delta(G_{\alpha\beta})}$ is 2-transitive of degree 3 or 4, contradicting (a).

Let p be a prime such that there is a nontrivial p-group fixing >2 points. Let $P \leqq G_{\alpha\beta}$ be such a p-group maximal with respect to fixing >2 points.

Lemma 2. $|\Delta(P)|=5$, $|N(P)^{\Delta(P)}|=20$ *and* $|\Delta(P) \cap \Gamma_i(\alpha, \beta)|=1$, $i=1, 2, 3$.

Proof. By Lemma 1, $N(P)^{\Delta(P)}$ is 2-transitive. Let $i=1, 2$, or 3. If a p-subgroup of $G_{\alpha\beta}$ fixes >1 point of $\Gamma_i(\alpha, \beta)$ it certainly fixes >2 points of Ω. Thus, if $\Delta(P) \cap \Gamma_i(\alpha, \beta) \neq \emptyset$ then by Lemma 1 $N(P)_{\alpha\beta}$ is transitive on $\Delta(P) \cap \Gamma_i(\alpha, \beta)$.

If $\Delta(P)$ meets all $\Gamma_i(\alpha, \beta)$ then (a) and (b) hold for $N(P)^{\Delta(P)}$, and the lemma follows from the minimality of n. Since $N(P)^{\Delta(P)}$ is not 3-transitive by (a), $\Delta(P)$ cannot meet just one $\Gamma_i(\alpha, \beta)$.

Suppose that $\Delta(P)$ meets just two sets $\Gamma_i(\alpha, \beta)$. There is a natural 1-1 correspondence between the orbits O of $N(P)^{\Delta(P)}$ of ordered triples (α, β, γ) of distinct points of $\Delta(P)$ and the orbits of $N(P)^{\Delta(P)}_{\alpha\beta}$ on $\Delta(P) - \{\alpha, \beta\}$. If O is such an orbit then so are $O' = \{(\alpha, \beta, \gamma) | (\beta, \gamma, \alpha) \in O\}$ and $O'' = \{(\alpha, \beta, \gamma) | (\gamma, \alpha, \beta) \in O\}$. Since two of O, O', O'' are the same in our case, $N(P)$ contains a 3-element $\neq 1$, which is not the case.

Lemma 3. *No nontrivial element fixes more than* 5 *points.*

Proof. If this is not the case, there is a prime p such that some nontrivial p-group fixes >5 points. Choose Q maximal among such p-groups. Set $\Delta = \Delta(Q)$ and $H = N(Q)^{\Delta}$. Let $P > Q$ be as in Lemma 2.

By Lemma 2, $|\Gamma_i(\alpha, \beta)| \equiv 1 \pmod p$, $i=1, 2, 3$. Thus, for any $\alpha^*, \beta^* \in \Delta$, $\alpha^* \neq \beta^*$, we have

$$|\Delta \cap \Gamma_i(\alpha^*, \beta^*)| \equiv 1 \pmod p), \quad i=1, 2, 3. \qquad (*)$$

If a p-subgroup of $G_{\alpha^*\beta^*}$ fixes >1 point of some $\Gamma_i(\alpha^*, \beta^*)$ then it fixes >5 points by Lemma 2. Thus, if $|\Delta \cap \Gamma_i(\alpha^*, \beta^*)|>1$ then Q is maximal among the p-subgroups of $G_{\alpha^*\beta^*}$ fixing >1 point of $\Gamma_i(\alpha^*, \beta^*)$. By Lemma 1, $H_{\alpha^*\beta^*}$ is transitive on $\Delta \cap \Gamma_i(\alpha^*, \beta^*)$, $i=1,2,3$. In particular, by the minimality of n, H is not 2-transitive on Δ. Frequent use will be made of the fact that, for any distinct $\alpha^*, \beta^* \in \Delta$, $H_{\alpha^*\beta^*}$ has precisely 5 orbits on Δ.

Let $\alpha \in \Delta$. By (*) H_α has a nontrivial orbit Φ on $\Delta - \alpha$. Let $\beta \in \Phi$.

We claim that H_α fixes no point $\delta \in \Delta - \alpha$. For otherwise, $H_{\alpha\beta} < H_\alpha = H_{\alpha\delta}$, where each of these groups has 5 orbits on Δ and β is an orbit of $H_{\alpha\beta}$ but not of $H_{\alpha\delta}$. This is clearly impossible.

In particular, H fixes no point $\delta \in \Delta$.

Suppose that H is intransitive on Δ. Let Δ' and Δ'' be (nontrivial) orbits of H on Δ. Let $\alpha' \in \Delta'$ and $\alpha'' \in \Delta''$. Both $\Delta' - \alpha'$ and $\Delta'' - \alpha''$ are unions of certain of the 5 orbits of $H_{\alpha'\alpha''}$ on Δ. By (*), $|\Delta' - \alpha'| \equiv 1$ or 2 and $|\Delta'' - \alpha''| \equiv 1$ or $2 \pmod p$, but $|\Delta' - \alpha'| \equiv |\Delta'' - \alpha''| \equiv 2 \pmod p$ does not occur. We may assume that $|\Delta'| \equiv 2 \pmod p$ and $H_{\alpha'\alpha''}$ is transitive on $\Delta' - \alpha'$. Then H is 2-transitive on Δ'. Let $\beta' \in \Delta' - \alpha'$. Note that $\Delta' \neq \{\alpha', \beta'\}$ since $H_{\alpha'}$ cannot fix the point $\beta' \in \Delta - \alpha'$. Thus $|\Delta'|>2$. Now $H_{\alpha'\beta'}$ has at least one orbit on Δ'' and hence at most two orbits on $\Delta' - \{\alpha', \beta'\}$. Consequently, $H^{\Delta'}$ is either 3-transitive or a transitive extension of a rank 3 group. As at the end of the proof of Lemma 2 we obtain $3 \mid |H|$, contradicting (a).

Thus, H is transitive on Δ. Recall that each orbit of $H_\alpha^{\Delta - \alpha}$ is nontrivial. Since H is not 2-transitive on Δ, we can find at least two orbits Φ, Φ' of H_α on $\Delta - \alpha$.

Here $|\Phi|$ and $|\Phi'|$ are >2. For suppose $|\Phi|=2$. Let $\alpha \neq \delta \in \Delta - \Phi$. By (*), $H_{\alpha\delta}$ fixes Φ pointwise, so $H_{\alpha\delta} \leqq H_{\alpha\beta}$. Since $H_{\alpha\delta}$ and $H_{\alpha\beta}$ have 5 orbits, $H_{\alpha\beta}$ must fix δ. Consequently, $H_{\alpha\beta}=1$, contradicting the fact that $|\Delta|>5$.

Both $\Phi - \beta$ and Φ' are unions of certain of the 5 orbits of $H_{\alpha\beta}$ on Δ. By (*), $|\Phi - \beta| \equiv 1$ or 2 and $|\Phi'| \equiv 1$ or $2 \pmod p$. Interchanging Φ and Φ' we find that either (i) $|\Phi| \equiv |\Phi'| \equiv 2 \pmod p$, $\Delta = \alpha \cup \Phi \cup \Phi'$, and H_α is 2-transitive on Φ and Φ'; or (ii) $p=2$, $|\Phi| \equiv |\Phi'| \equiv 1 \pmod 2$, $\Delta = \alpha \cup \Phi \cup \Phi'$, and H_α has rank 3 on Φ and Φ'.

Note that H is imprimitive on Δ. This follows from [9], Theorem 17.7 if (i) holds. If (ii) holds and $|\Phi'| \geqq |\Phi|$ then, since in this case $H_{\alpha\beta}$ is transitive on Φ', Φ' is an orbit of H_β and hence H_α is not maximal in H.

We may thus assume that the global stabilizer K of $\alpha \cup \Phi$ in H is transitive. Then $K^{\alpha \cup \Phi}$ is either 3-transitive or a transitive extension of a rank 3 group. Once again, as in the proof of Lemma 2 we obtain $3 \mid |K|$.

This contradiction proves the lemma.

We can now complete the proof of Theorem 2. Recall that each $|\Gamma_i(\alpha, \beta)|>1$.

For $i=1,2,3$, $G_{\alpha\beta}$ acts faithfully on $\Gamma_i(\alpha, \beta)$ as a regular or Frobenius group. To see this, let p be a prime and $x \in G_{\alpha\beta}$ a p-element fixing $\geqq 2$ points of some $\Gamma_i(\alpha, \beta)$. By Lemma 2, x fixes >5 points, so by Lemma 3 $x=1$.

Thus, $G_{\alpha\beta}$ has a unique normal subgroup A regular on each $\Gamma_i(\alpha, \beta)$. Here $|A|=(n-2)/3$.

If n is even so is $|A|$, and all involutions fix 0 or 2 points. By an elementary lemma of Hering [5, p. 164, (2)], $G_{\alpha\beta}$ has at most two orbits on $\Omega-\{\alpha,\beta\}$, contradicting (b).

Thus, n and $|A|$ are odd. Let $x=(\alpha\beta)\cdots$ be an involution. Then x normalizes A.

Suppose that $|G_{\alpha\beta}|$ is odd. Then x inverts A. If $\gamma^x=\gamma$ then x centralizes $G_{\alpha\beta\gamma}$. Thus, there are $|A|$ involutions $(\alpha\beta)\cdots$. Counting the pairs consisting of an involution and one of its 2-cycles shows that the set I of involutions of G_α has $|A|$ elements. Then $I\cup\{1\}$ is not a group, since $1+(n-2)/3 \nmid n-1$. By Bender's theorem [2], G has a nontrivial normal subgroup of odd order, and by the Feit-Thompson Theorem [3] G has a regular normal elementary abelian subgroup. This contradicts the semiregularity of A on $\Omega-\{\alpha,\beta\}$. (It is not difficult to replace Bender's theorem in the above argument by Bender's generalization of Burnside's theorem on permutation groups of prime degree [2], Lemma 2.5, according to which either G_α is 2-transitive on I or $G_\alpha= N(A)_\alpha\, C(I)$.)

Consequently, we may assume that $|\Delta(x)|=5$. Choose i such that $|\Delta(x)\cap\Gamma_i(\alpha,\beta)|>1$.

Since $C_A(x)$ is transitive on $\Delta(x)\cap\Gamma_i(\alpha,\beta)$ ([9], Theorem 11.2), by (a) we have $|C_A(x)|=5$, that is, $\Delta(x)\subseteq\Gamma_i(\alpha,\beta)$. Write $A=C_A(x)\times[A,x]$, where x inverts $[A,x]$.

Let $\gamma\in\Delta(x)$. Since x normalizes $G_{\alpha\beta\gamma}$ it centralizes some involution $y\in G_{\alpha\beta\gamma}$. Here y inverts A, so that $xy\in G_\gamma$ centralizes $[A,x]$. Since $\gamma^{[A,x]}\subseteq\Delta(xy)$ and xy is an involution, $|[A,x]|=1$ or 5, according to whether $|\Delta(xy)|=1$ or $\neq 1$. Consequently, $|A|=5$ or 25 and $n=17$ or 77.

It is easy to eliminate these possibilities by considering the index of the normalizer of a Sylow 17- or 19-subgroup. Alternatively, note that the pointwise stabilizer of $\Delta(x)$ is now a 2-group of order $\leqq 8$. Since this is normalized by a Sylow 5-subgroup F of $C(x)$ it follows that $F\lhd C(x)$. However, we have seen that corresponding to each 2-cycle $(\alpha\beta)$ of x there is a group of order 5 in $C(x)_{\alpha\beta}$. Thus, F fixes $\Omega-\Delta(x)$ pointwise, which is ridiculous.

This completes the proof of Theorem 2.

3. Proof of Theorem 1

As already remarked in § 1, we need only consider the cases $k=3$ and 4. G is $(k-1)$-transitive (Livingstone and Wagner [8], Theorem 2(a)).

Let $k=3$. Let Φ be a set of 3 points. The global stabilizer of Φ induces a permutation group on the ordered triples of distinct points of Φ each of whose orbits has the same length $6/f$. Here $f=2$, 3 or 6. Each orbit of G of ordered triples of distinct points has length $n(n-1)(n-2)/f$. Consequently, if $\alpha\neq\beta$ then each orbit of $G_{\alpha\beta}$ on $\Omega-\{\alpha,\beta\}$ has length $(n-2)/f$.

Suppose that $|G_\alpha|$ is odd. By a result of Bender [1], either (ii) holds or G is solvable. In the latter case, G has a regular normal elementary abelian subgroup N of order $n=2^d$. By [9], Theorem 10.4, $G_\alpha^{\Omega-\alpha}$ has a regular normal nilpotent subgroup M. Here M acts fixed-point-freely on N and hence is cyclic. Then $|N_{GL(d,2)}(M)|=(2^d-1)\,d$ implies that $2^d-2|6d$, so that (iii) holds.

We may thus assume that there is an involution of the form $(\alpha\beta)(\gamma)\cdots$. Then $f=3$, so that $3\nmid|G|$. Now Theorem 2 applies, whereas $n\geqq 2k=6$. This completes the proof when $k=3$.

Now let $k=4$. We first show that there is a set Φ of 4 points whose global stabilizer is transitive on Φ. By Livingstone and Wagner [8], Theorem 3, we can find a set Δ with $|\Delta|\geqq 4$ whose global stabilizer induces a 3-transitive group H on Δ such that each nontrivial element of H fixes $\leqq 3$ points of Δ. Certainly $4||H|$. If H has an element of order 4 we can find the desired Φ. If $|\Delta|\leqq 9$ our assertion is also clear. Finally, if $|\Delta|>9$ and if H contains a Klein group then this Klein group has an orbit of length 4.

It follows that G is transitive on the pairs (α,Φ) with $\alpha\in\Phi$ and $|\Phi|=4$. Then $G_\alpha^{\Omega-\alpha}$ is 3-homogeneous but not 3-transitive. If G_α is as in (ii) or (iii), then (iv) holds. This completes the proof of Theorem 1.

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