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Note

Locally Polar Lattices*

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Lattices are studied and characterized in which all intervals above points are polar spaces.

A lattice $\mathscr{L}$ is *locally polar* if each element is a join of points (atoms) and each interval $\mathscr{L}^x = \{y \mid x \leqslant y\}$, x a point, is a polar space (see Tits [6] or Buekenhout and Shult [4]). Recall that the *rank* of a polar space is the maximum projective dimension of an element; since this number is finite, $\mathscr{L}$ has a 1 and a 0. There is an obvious dimension function on $\mathscr{L}$, so lines and planes have the obvious meaning. We assume that $\mathscr{L}$ has the following properties.

(i) Each $\mathscr{L}^x$ has rank $n \geqslant 3$; if x and x' are distinct points, then $x \vee x'$ is 1 or a line.

(ii) If D, E, F, are planes such that $D \wedge E$ and $E \wedge F$ are lines, then there is a point x such that $x \vee D$, $x \vee E$ and $x \vee F$ are 3-spaces.

(iii) (Connectedness.) Given points p and q, there exist points $p = x_0, x_1, \ldots, x_k = q$ such that $x_{i-1} \vee x_i$ is a line for $i = 1, \ldots, k$.

(iv) Three pairwise collinear points are always coplanar.

THEOREM. *If $\mathscr{L}$ is a locally polar lattice satisfying* (i) *through* (iv), *then there is a canonical embedding of $\mathscr{L}$ into a polar space of rank $n + 1$.*

In particular, $\mathscr{L}$ can be canonically embedded in a projective space of dimension at most $2n + 3$, by the deep result of Tits [6].

This theorem has an obvious application to the program in Buekenhout [2]. Related results are found in Buekenhout [1] and Buekenhout and Hubaut [3]. Note, however, that our $\mathscr{L}$ need not be finite and that lines may have more than two points.

The proof of the theorem is a straightforward application of the method in [5]: we introduce ideal points and lines in a fairly natural way, and then

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appeal to the Buekenhout–Shult theorem [4]. After the proof, several examples are presented.

Notation. For $X \in \mathscr{L}$, dim X is one less than the minimum number of points with join X. For $X, Y \in \mathscr{L}$ we write $X \sim Y$ when $\dim X \vee Y = 1 + \max\{\dim X, \dim Y\}$. Thus, (ii) states that $x \sim D, E, F$.

Points are denoted by p, x, y, z, lines by K, L, M, N, X, Y, Z, and planes by D, E, F.

Abbreviate $\{L, M\}$ by LM whenever $L \sim M$.

Define $LM \circ LN$ whenever there is an $x \sim L \vee M, L \vee N$ with $(x \vee L) \wedge (x \vee M) = (x \vee L) \wedge (x \vee N)$; the latter element is then a line. Let $\equiv$ denote the equivalence relation generated by $\circ$ on the set of symbols LM.

The equivalence classes of $\equiv$ are called *ideal points* and denoted $\alpha, \beta, \gamma, \delta$, (Ideal lines are defined later.) If $LM \in \alpha$ we write $\alpha < L$ or $L > \alpha$. The most difficult part of the proof of the theorem is the following fact.

LEMMA 1. *Assume* (i) *and* (ii), *and let* $KL \in \alpha$.

(a) *If* $K \wedge L = p$, *then* α *consists of all pairs of coplanar lines on* p.
(b) *If* $K \wedge L = 0$, *then each point is on at most one line* $> \alpha$.

The proof of this lemma, and of the theorem itself, will be given in a sequence of steps.

(I) If $1 \neq S \in \mathscr{L}$ and $\dim S \geqslant 4$, then S is canonically embeddable in a projective space. In particular, if four lines of S have the property that five of the six pairs of lines are coplanar, then so is the sixth pair.

Proof. See [5].

(II) If $LM \circ LN$, and $y \sim L \vee M, L \vee N$, then $(y \vee L) \wedge (y \vee M) = (y \vee L) \wedge (y \vee N)$.

Proof. There is an $x \sim L \vee M, L \vee N$ with $(x \vee L) \wedge (x \vee M) = (x \vee L) \wedge (x \vee N) = X$ a line. Fix $p < L$. In the projective space underlying $\mathscr{L}^p$, the subspaces $L \vee M \vee x$ and $L \vee N \vee y$ have dimension 2, contain L, and hence span a subspace S of dimension at most 4 having L in its radical. Since $n \geqslant 3$, there is a point $p \vee z$ of $\mathscr{L}^p$ in $S^\perp$ on neither $L \vee M$ nor $L \vee N$. Let $Z = (z \vee L) \wedge (z \vee M)$. Then $Z < (L \vee M \vee x) \vee (p \vee z) \neq 1$ (since $L \vee M \vee x$ and $p \vee z$ are perpendicular in $\mathscr{L}^p$). By (I) applied to L, M, X, Z, we find that X and Z are coplanar. Then $Z = (z \vee L) \wedge (z \vee N)$ (by (I) applied to L, X, Z, N). Two further applications of (I) complete the proof.

(III) If $M_1M_2 \circ M_2M_3 \circ \cdots \circ M_rM_{r+1}$, $r \geqslant 3$, then there exists an M satisfying $M_1M_2 \circ M_1M \circ MM_{r+1}$ and $M \sim M_1 \vee M_2$.

Proof. Suppose $r = 3$. Then (ii) provides an $x \sim M_i \vee M_{i+1}$ for $i = 1, 2, 3$. By (II), $(x \vee M_i) \wedge (x \vee M_{i+1})$ is a line M independent of i. Then all requirements are met.

Now suppose $r > 3$. By induction, there is an N satisfying $M_2N \circ NM_{r+1}$. Now $M_1M_2 \circ M_2N \circ NM_{r+1}$, and we are back in the $r = 3$ case.

(IV) If $LM \equiv LN$ then $LM \circ LN$ or $M = N$.

Proof. By (III), there is a line X satisfying $ML \circ MX \circ XN$ and $X \sim L \vee M$. Use $x < X$ to establish $ML \circ LN$ if $M \neq N$.

(V) If $KL \equiv MN$, $K \wedge L = 0$, and $L, M > p$, then $L = M$.

Proof. By (III), there is an X with $KL \circ LX \circ XM$. Now use $x \sim K \vee L$, $L \vee X$, $X \vee M$ to complete the proof.

Proof of Lemma 1. Part (b) is just (V). Suppose $K \wedge L = p$ and $\alpha < M$. By (III), there exists N with $KL \circ LN \circ NM$. If $KL \circ LN$ via x then $(x \vee K) \wedge (x \vee L)$ must be $x \vee p$, from which $p < N$ follows. Similarly, $p < M$. Conversely, every line $M > p$ is coplanar with some line $N > p$ coplanar with L, and $KL \circ LN \circ NM$. This proves the lemma.

Definitions and Conventions. For α as in Lemma 1(a), we identify α with p. If $\alpha \neq p$ and there is a line $> \alpha, p$, this line is denoted $\alpha \vee p = p \vee \alpha$ and we write $p \sim \alpha$.

The *ideal line* $E \# F$ determined by distinct planes $E \sim F$ is defined by

$$E \# F = \{\alpha \mid \exists LM \in \alpha \text{ with } L < E, M < F\}.$$

We write $E \# F < E, F$ and $\alpha < E$, $\alpha < E \# F$ for α as in the definition. There is an obvious definition for *collinear* ideal points. According to (I), this is the "correct" definition inside $E \vee F$. As in the case of ideal points, we have to show that $E \# F$ can be equally well computed using other pairs of planes.

(VII) Suppose $E \sim F$, T is a 3-space $> E$, $p < T$, but $p \nless E$. Then there exists a plane D with $p < D < T$ and $E \# F = E \# D$.

Proof. Fix $x < E$. In $\mathscr{L}^x$, $E \vee F$ and T are planes on E. There is thus a 3-space $T' > E$ of $\mathscr{L}$ with $T' \sim E \vee F, T$. By (I), inside $T' \vee F$ there is a plane F' satisfying $F' < T'$ and $E \# F = E \# F'$. Similarly, there is a plane D with $p < D < T$ and $E \# F' = E \# D$. Similarly:

(VIII) If $\alpha < L < E$, then every point $p < E, p \nless L$, is on a line $N < E$ satisfying $LN \in \alpha$.

Proof. Let $LM \in \alpha$ and $x \sim E, L \vee M$. Set $X = (x \vee L) \wedge (x \vee M)$. Then $LX \in \alpha$. Now (I) applies (within any 4-space $> x \vee E = X \vee E$).

(IX) Ideal points α, β are collinear iff $\alpha < L, \beta < M$ for some coplanar lines L, M.

Proof. If $\alpha, \beta < E \# F$, then the desired lines can be found in E.

Conversely, assume $\alpha < L, \beta < M$, and $L \vee M = E$ is a plane. By (VIII), $LL' \in \alpha$ and $MM' \in \beta$ for some $L', M' < E$. Let $x \sim E$. Then $X = (x \vee L) \wedge (x \vee L')$ and $Y = (x \vee M) \wedge (x \vee M')$ are lines by (II), and both are on $x \vee E$. Thus, $F = X \vee Y$ is a plane, and $\alpha, \beta < E \# F$.

LEMMA 2. *Assume* (i) through (iii). *If* α *is an ideal point and* Λ *an ideal line, then* α *is collinear with some* $\beta < \Lambda$.

(X) For any p and α there is a point $q \sim p, \alpha$.

Proof. Let $K_0, K_1, \ldots, K_n$ be lines with $\alpha < K_0, p < K_n$, and each $K_i \wedge K_{i+1}$ a point. Assume $n \geqslant 3$, and consider K_0, K_1, K_2, K_3. There is a line $L_1 > K_0 \wedge K_1$ with $L_1 \sim K_0, K_1$. Now we have $K_2, K_1 \vee L_1 > K_1 \wedge K_2$, so there is a line L_2 coplanar with K_2 and satisfying $K_1 \wedge K_2 < L_2 < K_1 \vee L_1$.

By (ii), there is a point $x \sim K_0 \vee L_1, L_1 \vee L_2, L_2 \vee K_2$. In particular, $x \sim K_1 \wedge K_2$ and α by (IX). Now set $K_1' = x \vee \alpha$ and $K_2' = x \vee (K_1 \wedge K_2)$ and decrease n.

If $n = 2$, the same argument applies, this time with $x \sim p$ and α.

Proof of Lemma 2. We are given α and $\Lambda = E \# E'$. Pick $p < E$. By (*X*), there is a point $q \sim \alpha, p$. As above, there are planes F_1, F_2 such that $E \wedge F_1$ is a line, $p \vee q < F_1, q \vee \alpha < F_2$, and $F_1 \wedge F_2$ is a line. By (ii), there is an $x \sim E, F_1, F_2$. Then $x \sim \alpha$, and by (VII) there is a plane E'' with $E \# E' = E \# E''$ and $x < E'' < x \vee E$. Now $x \vee \alpha$ is coplanar with some line L satisfying $x < L < E''$. By (I), L and $E \# E''$ are on a common ideal point β. Since $\beta < E \# E''$ and $(x \vee \alpha) \vee (x \vee \beta)$ is a plane, this proves the lemma by (IX).

LEMMA 3. *Assume* (i) *through* (iv). *If* α *is collinear with two ideal points* β, γ *of an ideal line* Λ, *then* α *is collinear with every ideal point of* Λ.

Axiom (iv) is used as follows.

(XI) Suppose $\alpha < N_i$, and $L \wedge N_i = p_i$ is a point, for $i = 1, 2$. Then $N_1 \sim N_2$.

Proof. By (III), there is a line $N > \alpha$ coplanar with both N_1 and N_2. Let $x < N$. Then $x \vee p_1 \vee p_2$ is a plane by (iv), and $x \vee p_i$ is perpendicular to $N \vee N_{3-i}$ in $\mathscr{L}^x$. Thus, $x \vee p_2 \vee N_1$ is a 3-space T containing $N \vee p_1 \vee p_2$. It follows that $N, N_1, N_2 < T$, and hence that $N_1 \vee N_2$ is a plane by (I).

(XII) Let α, β, Λ be as in the lemma, with $\Lambda = E \# E'$. Then there is a point $x \sim \alpha, E$ with $x \vee \alpha \sim x \vee \beta$.

Proof. By (III) and (IX), there are lines $L_1 \sim L_2 \sim L_3 \sim L_4$ with $\beta < L_1 < E, \beta < L_3, \alpha < L_4$. Let $y \sim L_1 \vee L_2, L_2 \vee L_3, L_3 \vee L_4$. By (VIII), $y \sim \alpha$. By Lemma 1 and (VIII), $y \vee \beta$ exists and is coplanar with L_1. Since $y \sim L_3 \vee L_4$, we have $y \vee \alpha \sim y \vee \beta$. Any $x \sim E, L_1 \vee (y \vee \beta), (y \vee \beta) \vee (y \vee \alpha)$ meets our requirements.

Proof of Lemma 3. Let $x \sim \alpha, E$ with $x \vee \alpha \sim x \vee \beta$. By (VII), we may assume $x < E' < E \vee x$. Then let $y \sim \alpha, E'$ with $y \vee \alpha \sim y \vee \gamma$. We may assume $y < E < E' \vee y$. Then $y \vee \beta, y \vee \gamma < E$ by (VIII). We can apply (XI) to $L = x \vee y, N_1 = x \vee \alpha$, $N_2 = y \vee \alpha$ and obtain $x \vee \alpha \sim y \vee \alpha$. Then $x \vee y \sim x \vee \alpha$, so $(x \vee y) \vee (x \vee \alpha) \vee (x \vee \beta)$ is a 3-space containing $y \vee \alpha$ and $y \vee \beta$. Thus, $y \vee \alpha \sim y \vee \beta$. We already know $y \vee \alpha \sim y \vee \gamma$. Thus, using $\mathscr{L}^y$ we see $y \vee \alpha \sim E$. Then $y \vee \alpha \sim y \vee \lambda$ for all $\lambda < E \# E' = \Lambda$, as required (see (IX)).

Proof of the Theorem. Let α be an ideal point, $\alpha < L$, and $\alpha \neq p < L$. Pick any $q \sim p$ not coplanar with L. Then q and α cannot be collinear by (XI). In view of Lemmas 2 and 3, the Buekenhout–Shult theorem [4] now completes the proof.

EXAMPLES. A lattice associated with the simple group F_{22} (see Buekenhout and Hubaut [3]) has $n = 2$ in (i): $\mathscr{L}^x$ is of type $SU(6, 2)$. In this case, (I) even fails: The points and planes in a 3-space form a Steiner system $S(22, 6, 1)$. However, for the case of $SU(7, K)$, $O(7, K)$ and $O^-(8, K)$, assuming (ii) through (iv) the theorem should still be true.

That (i), (iii), and (iv) do not imply (ii) is seen from the following examples. Let V be an orthogonal or unitary vector space, and let $\mathscr{L}$ consist of $\varnothing$, V, the vectors in V, and the translates of all totally singular subspaces. Of course, one can also delete some such points and subspaces and still arrange to have a locally polar lattice. If instead V had been chosen to be symplectic, then (ii) and (iv) would both fail.

Examples satisfying (ii), (iii), usually (i), but not (iv), are constructed as follows. Let K be a field of characteristic 2, V a nondegenerate orthogonal vector space over K, and R a nonsingular 1-space. Then $\mathscr{L}$ consists of ($\varnothing$, 1, and) all $(X + R)/R = \bar{X}$ for X a totally singular subspace not contained in $R^\perp$. If $W \nleq R^\perp$ is a singular 1-space, then $\mathscr{L}^{\bar{W}}$ is clearly just the polar space for $W^\perp/W$. Thus, $\mathscr{L}$ is locally polar. If K is perfect, an easy computation shows that any two points are collinear! Since any K contains $GF(2)$, it follows that (iv) fails. Note that, if $K \neq GF(2)$, then one can again delete some elements of $\mathscr{L}$ and still arrange to have a locally polar lattice. Lemmas 1 and 2 hold here, the ideal points being those of $\mathscr{L}$ along with all 1-spaces of $R^\perp/R$.

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