ON THE MAXIMALITY OF PSL($d+1, q$), $d \geqslant 2$

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It is well known that the Mathieu groups M_{12} and M_{24} contain PSL(2, 11) and PSL(2, 23) in their natural permutation representations of degrees 12 and 24, respectively (see, e.g., [**2**]). It is therefore natural to consider the permutation representation of $H = \mathrm{PSL}(d+1, q)$ of degree $v = (q^{d+1}-1)/(q-1)$ on the set Ω of points of projective d-space $PG(d, q)$, and to ask what subgroups of Sym (Ω) contain H. While nothing seems to be known for $d = 1$, we shall answer this question when $d \geqslant 2$.

THEOREM. *Suppose $d \geqslant 2$, and let G be a subgroup of* Sym (Ω) *containing H. Then either* (i) *G is contained in the normalizer* $\mathrm{P\Gamma L}(d+1, q)$ *of H, or* (ii) Alt (Ω) $\leqslant G$.

Proof. G is clearly 2-transitive on Ω. A hyperplane Φ of $PG(d, q)$ has $k = (q^d-1)/(q-1)$ points. Suppose first that G is not k-transitive. Note that the pointwise stabilizer of Φ is transitive on $\Omega - \Phi$, the set stabilizer of Φ is 2-transitive on $\Omega - \Phi$, and $|\Omega - \Phi| = q^d$ is a prime power. By [**4**; Theorem 8.4], or by [**5**] and [**3**; Theorem 9], one of the following must hold: (a) G is a group of collineations of $PG(d', q')$, for some d' and q', and Φ is a hyperplane of $PG(d', q')$; (b) v is a power of 2 and $k = v/2$; or (c) $v = 22$, 23 or 24. Both (b) and (c) are excluded for arithmetical reasons. (a) implies that $d' = d$, $q' = q$ and $G \leqslant \mathrm{P\Gamma L}(d+1, q)$ (see, e.g., [**1**; Theorem 4, p. 88]). Thus (i) must hold in this case.

Now suppose that G is k-transitive on Ω. If Alt (Ω) $\nleqslant G$ then $k < 3 \log (v-k)$ by [**7**; Satz C] (compare [**8**; p. 21]). There are only nine possible pairs v, k which satisfy this inequality. Suitable applications of [**8**; Theorems (13.9) and (13.11)] show that in these cases, as for the values of v and k not satisfying the inequality, we must have Alt (Ω) $\leqslant G$.

Remark. In view of a result of Tits (see [**6**; Lemma (1.6)]), one obtains the same result for any transitive representation of PSL($d+1, q$) of degree $(q^{d+1}-1)/(q-1)$.

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