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Translation Planes of Order q^6 Admitting $SL(2, q^2)$

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Large numbers of translation planes are constructed which have order q^6 and admit a collineation group $SL(2, q^2)$ generated by elations.

In this paper we will give a simple construction for at least $q(q+1)/6e$ nondesarguesian translation planes $\mathcal{A}$ of order q^6 whenever $q = p^e$ with p a prime. The elations of $\mathcal{A}$ fixing the origin 0 generate a group $S = SL(2, q^2)$ having $q+2$ orbits on the line L_∞ at infinity. The group $(\mathrm{Aut}\,\mathcal{A})_0$ has just two orbits on L_∞, of lengths q^2+1 and $q^6 - q^2$. The kernel of $\mathcal{A}$ is $GF(q^3)$, and S acts irreducibly on the underlying four-dimensional $GF(q^3)$-space exactly as it does in the case of the desarguesian plane of order q^6.

The construction was motivated by Example 8.2 of [2]. However, the plane of order 2^6 constructed there is not the same as the one obtained here. Variations on the construction are undoubtedly possible.

The planes also differ from those in [1]: the kernel and action on L_∞ are quite different for those planes.

THEOREM. *Let $q = p^e > 1$ be a power of a prime p. Then there are at least $q(q+1)/6e$ different nondesarguesian translation planes $\mathcal{A}$ of order q^6 having kernel $GF(q^3)$ such that $G = (\mathrm{Aut}\,\mathcal{A})_0 \cap GL(4, q^3)$ behaves as follows.*

(i) *G has orbits of lengths q^2+1 and $q^6 - q^2$ on L_∞.*

(ii) *$G \trianglerighteq S \cong SL(2, q^2)$, where S has one orbit on L_∞ of length q^2+1 and $q+1$ of length $(q^2+1)\, q^2(q-1)$.*

(iii) *S fixes q^2+q+1 desarguesian subplanes of order q^2 containing 0 which are permuted transitively by the homologies of $\mathcal{A}$ with center 0.*

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(iv) *Each Sylow p-subgroup of G consists of q^2 elations with the same axis.*

(v) $G = (GF(q^3)^* GL(2, q^2)) \cdot \mathbb{Z}_2$.

(vi) $G \leqslant \Gamma L(2, q^6)$, *and G acts on the four-dimensional $GF(q^3)$-space underlying $\mathcal{A}$ exactly as it does for the desarguesian plane of order q^6; in particular, S acts irreducibly over $GF(q^3)$, while G acts irreducibly over $GF(p)$.*

Proof. Set $K = GF(q)$ and $F = GF(q^3)$. Let V be an $\Omega^+(6, q)$ space with quadratic form Q and bilinear form (,). Set $V^F = V \otimes_K F$, and extend Q and (,) to forms Q^F and $(\ ,\ )^F$ on V^F. (Thus, there is a basis $v_1, \ldots, v_6$ of V such that $Q^F(\sum \alpha_i v_i) = \alpha_1\alpha_6 + \alpha_2\alpha_5 + \alpha_3\alpha_4$ for all $\alpha_i \in F$.)

Under the Klein correspondence, the singular points of V^F correspond to the lines of $PG(3, q^3)$. A spread of $PG(3, q^3)$ can be obtained from a set of $q^6 + 1$ singular points of V^F no two of which are perpendicular.

Fix an $\Omega^-(4, q)$ subspace W of V. Let E be any set of $q^2 + 1$ singular vectors in W such that $\{\langle e\rangle \mid e \in E\}$ consists of all singular points of W. No two members of E are perpendicular.

Let N be a set of $q + 1$ vectors in $W^\perp$ no two of which are linearly dependent.

If $\alpha \in F$, let $\alpha^{1/2}$ denote a square root of α, if one exists.

Fix $\gamma \in F - K$.

Let Ω_γ consist of the following points of V^F (where $e, f \in E$, $e \neq f$, $k \in K^*$ and $n \in N$):

$$\langle e\rangle,$$
$$\langle e + k\gamma f \pm [k\gamma(e, f)/Q(n)]^{1/2} n\rangle.$$

We will show that Ω_γ consists of $q^6 + 1$ pairwise nonperpendicular singular points of V^F.

Each of these points is easily checked to be singular.

If q is odd, fix e, f and n. Then $K^*\gamma(e,f)/Q(n)$ contains exactly $\frac{1}{2}(q-1)$ squares. Thus, $|\Omega_\gamma| = (q^2 + 1) + (q^2 + 1)q^2(q + 1) \cdot \frac{1}{2}(q - 1) \cdot 2 = q^6 + 1$. Similarly, $|\Omega_\gamma| = q^6 + 1$ if q is even.

Let $e', f' \in E$, $e' \neq f'$, $k' \in K^*$ and $n' \in N$. Note that

$$(e', e + k\gamma f \pm [k\gamma(e, f)/Q(n)]^{1/2} n)^F = (e', e) + k(e', f)\gamma \neq 0$$

since $(e', e) \neq 0$ or $(e', f) \neq 0$ (as $e \neq f$), while $\gamma \notin K$. Suppose that

$$\begin{aligned} 0 = (e + k\gamma f &\pm [k\gamma(e, f)/Q(n)]^{1/2} n, \\ &e' + k'\gamma f' \pm [k'\gamma(e', f')/Q(n')]^{1/2} n')^F \\ &= (e, e') + l\gamma + kk'(f, f')\gamma^2, \end{aligned}$$

where

$$l = k'(e, f') + k(e', f)$$
$$\pm \gamma^{-1}[k\gamma(e, f)/Q(n)]^{1/2}[k'\gamma(e', f')/Q(n')]^{1/2}(n, n') \in K$$

(as $a\gamma$ is a square for some $a \in K$). Since γ is cubic over K, it follows that $e = e', f = f'$ and $l = 0$. In view of the definition of l,

$$Q^F([k\gamma(e, f)/Q(n)]^{1/2}n \pm [k'\gamma(e, f)/Q(n')]^{1/2}n') = \gamma l = 0.$$

Since $(W^F)^\perp$ is anisotropic,

$$[k\gamma(e, f)/Q(n)]^{1/2}n \pm [k'\gamma(e, f)/Q(n')]^{1/2}n' = 0.$$

In view of the definition of N, it follows that $n = n'$ and $k\gamma(e, f)/Q(n) = k'\gamma(e, f)/Q(n)$. Thus, our original two vectors are one.

This shows that Ω_γ determines a translation plane $\mathcal{A}_\gamma$ of order q^6. Since Ω_γ spans V^F, $\mathcal{A}_\gamma$ is nondesarguesian. Its kernel is then $GF(q^3)$.

If some members of N are replaced by nonzero scalar multiples of themselves, the definition of Ω_γ produces the same set Ω_γ. Similarly, since

$$le + k\gamma f \pm [k\gamma(le, f)/Q(n)]^{1/2}n$$
$$= l\{e + k'\gamma f \pm [k'(e, f)/Q(n)]^{1/2}n\}$$

whenever $k = lk'$ and $l \in K^*$, different choices for E produce the same set Ω_γ. Consequently, Ω_γ is invariant under the group J of all $g \in GL(6, q)$ such that $W^g = W$ and $Q(v^g) = c_g Q(v)$ for all $v \in V$ and some $c_g \in K$. Here, J induces a group of collineations and correlations of $PG(3, q^3)$. Let H be the subgroup of index 2 of J inducing collineations of $PG(3, q^3)$.

Note that $H > \Omega^-(4, q) \times \Omega^-(2, q)$, where $\Omega^-(4, q) \cong PSL(2, q^2)$. A Sylow p-subgroup of $\Omega^-(4, q)$ fixes some $e \in E$ and induces the identity on the F-space $e^\perp/\langle e \rangle$. This proves (iv), and implies that G has an orbit of length $q^2 + 1$ on L_∞. It is now straightforward to check that (i)–(vi) hold (since G and H agree on Ω_γ while H fixes W^F and hence induces a collineation group of $AG(2, q^6)$).

Let γ, $\delta \in F - K$. An isomorphism from $\mathcal{A}_\gamma$ to $\mathcal{A}_\delta$ induces a transformation $g \in \Gamma L(6, q^3)$ such that $(\Omega_\gamma)^g = \Omega_\delta$ and $Q(v^g) = c_g Q(v)^\sigma$ for all $v \in V^F$ and some $c_g \in F$, $\sigma \in \operatorname{Aut} F$. Then $(W^F)^g = W^F$ by (i). Projecting Ω_γ and Ω_δ onto W^F shows that $W^g = W$. Using H, we can modify g in order to have g induce a field automorphism σ on W and hence on V. Now $(\Omega_\gamma)^g = \Omega_{\gamma^\sigma}$ by definition. Thus, $\Omega_\delta = \Omega_{\gamma^\sigma}$. It follows that $\delta = k(\gamma^{\pm 1})^\sigma$ for some $k \in K^*$. The isomorphism classes of planes $\mathcal{A}_\gamma$ therefore correspond to orbits on $F - K$ of the group of permutations $\gamma \to k(\gamma^{\pm 1})^\sigma$ with $k \in K^*$ and $\sigma \in \operatorname{Aut} F$. Each orbit has length $\leqslant (q-1) \cdot 2 \cdot 3e$. Thus, there are at least $(q^3 - q)/(q-1)6e$ different planes. This completes the proof of the theorem.

Remarks. (1) The $\mathbb{Z}_2$ in (v) induces a Baer involution of $\mathcal{Q}_\gamma$.

(2) $\Omega^-(2,q)=\mathbb{Z}_{q+1}$ fixes the point 0 and $q+1$ points of L_∞, but is not planar since $GF(q^3)^*$ centralizes it.

(3) Since a cyclic subgroup of S order $q+1$ fixes exactly $2+(q+1)\cdot 2(q-1)$ points of L_∞, it is not planar. However, there are desarguesian subplanes of order q^2 on which this cyclic group induces $q+1$ homologies.

(4) The $\mathcal{Q}_\gamma$ are the only translation planes of order q^6 with kernel $GF(q^3)$ admitting a group $GL(2,q^2)$ whose representation on the underlying vector space is as in the theorem. In order to see this, observe that such a plane is again represented by a set Ω of q^6+1 points in our orthogonal geometry V^F. Moreover, $GL(2,q^2)$ corresponds to our group H, and leaves invariant $GF(q)$-spaces W and $W^\perp$ as before. A search for orbits of at most q^6+1 singular points yields $\Omega=\Omega_\gamma$ for some γ.

References

1. W. M. Kantor, Spreads, translation planes and Kerdock sets, I, *SIAM J. Alg. Disc. Methods*, in press.
2. W. M. Kantor, Ovoids and translation planes, to appear.

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