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Symplectic Groups, Symmetric Designs, and Line Ovals*

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1. INTRODUCTION

Let $\Pi = Sp(2m, 2)$ and $\Gamma = \Sigma\Pi$, where Σ is the translation group of the affine space $AG(2m, 2)$. Π acts 2-transitively on the cosets of each orthogonal subgroup $GO^{\epsilon}(2m, 2)$, $\epsilon = \pm 1$, and Γ has a second class of subgroups isomorphic to Π ([10, pp. 236, 240], [6], and [14]). By considering a certain symmetric design $\mathscr{S}^{\epsilon}(2m)$ having Γ as its full automorphism group, we will prove these results. The symmetric designs $\mathscr{S}^{\epsilon}(2m)$ will be studied and characterized in terms of an interesting property concerning the symmetric difference of distinct blocks. Other properties and characterizations of these desings will also be given, and an application made to rank 3 linear groups.

There are several ways to construct $\mathscr{S}^{\epsilon}(2m)$. One way is in terms of difference sets [3, p. 108]; from this point of view, $\mathscr{S}^{\epsilon}(2m)$ has the unusual property of arising from difference sets in $m + 1$ nonisomorphic abelian groups. Using a description in terms of the incidence matrix, Block [1] observed that the automorphism group of $\mathscr{S}^{\epsilon}(2m)$ is 2-transitive. The present work was motivated by Block's result. In the course of studying the full automorphism group $\mathscr{S}^{\epsilon}(2m)$, a description was found in terms of $GO^{\epsilon}(2m, 2)$ and $Sp(2m, 2)$ (see Section 4). More recently, the designs $\mathscr{S}^{\epsilon}(2m)$ were discovered in terms of the latter description by A. Rudvalis (unpublished) and Cameron and Seidel [2].

Yet another description of the designs $\mathscr{S}^{1}(2m)$ arises from using suitable dual ovals in desarguesian and Lüneburg–Tits planes. The construction in terms of desarguesian planes was obtained by the author [3, p. 95] at the same time that the incidence matrix and symplectic group descriptions were first considered; however, it was not originally known that the descriptions produced the same designs. The construction from the Lüneburg–Tits planes requires the use of the designs to prove a new property of the planes. We note

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that, using suitable dual ovals, new symmetric designs can also be obtained having the same parameters as $\mathscr{S}^1(2m)$. The determination of those translation planes which are related to $\mathscr{S}^1(2m)$ as in Section 7 seems to be an interesting but difficult problem.

2. Definitions

The definitions and elementary properties of symplectic and orthogonal groups are assumed from [5]. $GO^\epsilon(2m, 2)$ will denote $O_{2m}(F_2, Q)$ in Dieudonne's notation, where $\epsilon = +1$ if the quadratic form Q has index m, and $\epsilon = -1$ if this index is $m - 1$.

The basic properties of symmetric designs and difference set designs are found in [3]. Isomorphic designs will be identified. Symmetric designs will always be assumed to have $v \geqslant k + 2 \geqslant 4$, except for the case $v = 4$ and $k = 1$ or 3.

Points will be denoted by 0, x, y, z and blocks by B, C, D, E, X, Y.

If S and T are sets of points of a design, then $\mathscr{C}S$ is the complement of S and $S \triangle T$ is the symmetric difference $(S \cup T) - (S \cap T)$. (This notation is intended to distinguish the symmetric difference of blocks from the sum of blocks, which will be defined later in a special situation.)

If Γ is a permutation group, Γ_S denoted the global stabilizer of the set S.

3. The Designs $\mathscr{S}^\epsilon(2m)$

Set

$$H(2) = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

For each positive integer m. Let $H(2m)$ be the tensor product of m copies of $H(2)$. Rows and columns of $H(2m)$ have $2^{2m-1} + \epsilon 2^{m-1}$ entries ϵ, where $\epsilon = \pm 1$. Any two distinct rows or columns have in common $2^{2m-2} + \epsilon 2^{m-1}$ entries ϵ. We can thus regard rows and columns as the points and blocks of a symmetric design $\mathscr{S}^\epsilon(2m)$, a point being on a block if and only if the corresponding entry is ϵ. $\mathscr{S}^\epsilon(2m)$ has parameters

$$v = 2^{2m}, \qquad k = 2^{2m-1} + \epsilon 2^{m-1}, \qquad \lambda = 2^{2m-2} + \epsilon 2^{m-1}. \tag{1}$$

$\mathscr{S}^1(2m)$ and $\mathscr{S}^{-1}(2m)$ are complementary designs. $\mathscr{S}^1(2m)$ was first described in the present form by Block [1].

As $H(2m)$ is symmetric, $\mathscr{S}^{\epsilon}(2m)$ is self-dual.

$\mathscr{S}^{\epsilon}(2m)$ has the following basic property:

LEMMA 1. *Points can be labelled by the elements of an additive group G in such a way that $\Sigma = \{x \to x + g \mid g \in G\}$ is an automorphism group sharply transitive on the points, and, for any distinct blocks X and Y, $X \triangle Y$ is a left coset of a subgroup of G.*

Proof. As $X \triangle Y = \mathscr{C}X \triangle \mathscr{C}Y$, we may assume that $\epsilon = 1$. If $m = 1$, then $\mathscr{S}^1(2)$ has a unique automorphism group Σ_2 with the stated properties. If $m > 1$, let Σ_{2m-2} be an automorphism group of $\mathscr{S}^1(2m-2)$ which satisfies Lemma 1. The elements of Σ_2 and Σ_{2m-2} may be regarded as pairs of permutation matrices. By taking the tensor products of such matrices we find that $\Sigma = \Sigma_2 \times \Sigma_{2m-2}$ is an automorphism group of $\mathscr{S}^1(2m)$ sharply transitive on the points. Let G_2 and G_{2m-2} correspond to Σ_2 and Σ_{2m-2} as in Lemma 1. As the rows of $H(2)$ and $H(2m-2)$ are labelled by G_2 and G_{2m-2}, the rows of $H(2m)$ can be labelled by $G = G_2 \times G_{2m-2}$ so that $\Sigma = \{x \to x + g \mid g \in G\}$. Regard G as the set of ordered pairs (x_2, x_{2m-2}) with $x_2 \in G_2$ and $x_{2m-2} \in G_{2m-2}$. Then there is a 1–1 correspondence between the blocks B of $\mathscr{S}^1(2m)$ and the pairs B_2, B_{2m-2} of blocks of $\mathscr{S}^1(2)$ and $\mathscr{S}^1(2m-2)$, respectively, such that

$$B = (B_2, B_{2m-2}) \cup (\mathscr{C}B_2, \mathscr{C}B_{2m-2}).$$

Suppose also that

$$B \neq X = (X_2, X_{2m-2}) \cup (\mathscr{C}X_2, \mathscr{C}X_{2m-2}).$$

Then

$$B \triangle X = (B_2 \triangle X_2, B_{2m-2} \triangle X_{2m-2}) \cup (\mathscr{C}(B_2 \triangle X_2), \mathscr{C}(B_{2m-2} \triangle X_{2m-2})).$$

$B_2 \triangle X_2$ is empty or a coset of a subgroup H_2 of G_2. By induction, $B_{2m-2} \triangle X_{2m-2}$ is empty or a coset of a subgroup H_{2m-2} of G_{2m-2} of order 2^{2m-3}. If either $B_2 \triangle X_2$ or $B_{2m-2} \triangle X_{2m-2} = \varnothing$, our assertion is immediate. If both are nonempty, write $B_2 \triangle X_2 = x_2 + H_2$, $\mathscr{C}(B_2 \triangle X_2) = y_2 + H_2$, $B_{2m-2} \triangle X_{2m-2} = x_{2m-2} + H_{2m-2}$ and $\mathscr{C}(B_{2m-2} \triangle X_{2m-2}) = y_{2m-2} + H_{2m-2}$. Then $B \triangle X = (x_2, x_{2m-2}) + H$, where H is the subgroup generated by (H_2, H_{2m-2}) and $(x_2 + y_2, x_{2m-2} + y_{2m-2})$. This proves Lemma 1.

Note that G is an elementary abelian 2-group in Lemma 1. However, in the first part of the above proof we could have taken Σ_2 to be any one of the four sharply transitive subgroups of S_4. Thus, $\mathscr{S}^{\epsilon}(2m)$ may be regarded as a difference set design in many ways, of which at least $m + 1$ are inequivalent from the point of view of difference sets. For $\epsilon = -1$, these difference sets are described on p. 108 of [3].

Note also that the dual of Lemma 1 holds with the same group Σ. This follows from the fact that all the permutation matrices involved are involutory and hence symmetric.

4. Properties and Characterizations of $\mathscr{S}^\epsilon(2m)$

Let $\mathscr{D}$ be a symmetric design satisfying Lemma 1. We will prove properties of $\mathscr{D}$ which will yield the automorphism group of $\mathscr{S}^\epsilon(2m)$ together with characterizations of $\mathscr{S}^\epsilon(2m)$. As usual, set $n = k - \lambda$. Fix a block B on the identity element o of G. If $X \neq B$, set

$$H_X = \{Y \mid |(B \triangle X) \cap Y| = n\} = \{B, X\} \cup \{Y \mid |B \cap X \cap Y| = \lambda - \tfrac{1}{2}n\}.$$

We first show that $\Sigma_{B \triangle X}$ is transitive on H_X and that Σ is an elementary abelian 2-group. As $\mathscr{C}X \triangle \mathscr{C}Y = X \triangle Y$, we may temporarily assume that $k < 2n$. $\Sigma_{B \triangle X}$ is a group of order $|B \triangle X| = 2n$, so that $|H_X| = 2an$ for some integer a. Since $\Sigma_{B \triangle X}$ is transitive on $B \triangle X$, each point of $B \triangle X$ is on $2an \cdot n/2n = an$ blocks of H_X. As $k < 2n$, $a = 1$ and $\Sigma_{B \triangle X}$ is transitive on H_X. Let $1 \neq \sigma \in \Sigma$ and set $X = B^\sigma$. If $\sigma' \in \Sigma_{B \triangle X}$ moves B to X, then $\sigma = \sigma'$. Thus, $B \triangle X = (B \triangle X)^\sigma = X \triangle X^\sigma$, so $B = X^\sigma$. It follows that Σ has exponent 2. By a result of Mann [3, p. 61], $\mathscr{D}$ has parameters (1). In particular, $2n = \frac{1}{2}v$.

Σ determines affine spaces $\mathfrak{A}$ and $\mathfrak{A}^\#$ over $GF(2)$, each with translation group Σ, such that the points of $\mathfrak{A}$ and $\mathfrak{A}^\#$ are, respectively, the points and blocks of $\mathscr{D}$. H_X is a hyperplane of $\mathfrak{A}^\#$. If B is regarded as the "origin" in $\mathfrak{A}^\#$, the sum of two blocks is meaningful. Moreover, the definition of H_X shows that $X \to H_X$, $X \neq B$, defines a symplectic polarity of the projective space $\mathfrak{A}^\# - \{B\}$. Let $f(X, Y)$ be the corresponding bilinear form.

The definition of addition shows that, if $Y \in H_X$ and $\sigma \in \Sigma_{B \triangle X}$ takes B to Y, then $X^\sigma = X + Y$. Thus, $B \triangle X = Y \triangle (X + Y)$. If $Y \notin H_X$ and $\sigma \in \Sigma$ takes B to Y, then $\mathscr{C}(B \triangle X) = (B \triangle X)^\sigma = Y \triangle (X + Y)$. This proves:

Lemma 2. *If $X \neq B$ and Y are blocks, then $B \triangle X \triangle Y = Y + X$ or $\mathscr{C}(X + Y)$ according to whether $Y \in H_X$ or $Y \notin H_X$.*

For each block X let $Q(X) \in GF(2)$ be 0 if $o \in X$ and 1 if $o \notin X$. Then $Q(B) = 0$. We claim that, for all blocks X and Y,

$$Q(X + Y) = Q(X) + Q(Y) + f(X, Y). \tag{2}$$

Repeated use will be made of Lemma 2. We may assume that $X \neq B$, $Y \neq B$, $X \neq Y$. Let $Q(X) = 0$. Then $o \notin B \triangle X$. If $Y \in H_X$ then $f(X, Y) = 0$,

$B \triangle X = Y \triangle (X + Y)$, and thus $Q(Y) = 0$ if and only if $Q(X + Y) = 0$. If $Y \notin H_X$ then $f(X, Y) = 1$, $B \triangle X = \mathscr{C}(Y \triangle (X + Y))$, and thus $Q(Y) = 0$ if and only if $Q(X + Y) = 1$.

Next, let $o \notin X, Y$. Then $o \in B \triangle X$ and $Q(X) = Q(Y) = 1$. If $Y \in H_X$ then $f(X, Y) = 0$, $B \triangle X = Y \triangle (X + Y)$, and thus $o \in X + Y$, so $Q(X + Y) = 0$. Finally, if $Y \notin H_X$, then $f(X, Y) = 1$,

$$B \triangle X = \mathscr{C}(Y \triangle (X + Y)),$$

and thus $o \notin X + Y$, so $Q(X + Y) = 1$. This proves (2).

Thus, Q is a quadratic form associated with f. The set of blocks on o is its set of singular points. As $k = 2^{m-1}(2^m + \epsilon)$, Q has index m if $\epsilon = 1$ or $m - 1$ if $\epsilon = -1$. This proves that $\mathscr{D}$ is unique, and thus is $\mathscr{S}^\epsilon(2m)$. We can now prove the following:

THEOREM 1. *The full automorphism group Γ of $\mathscr{S}^\epsilon(2m)$ is a semidirect product of the translation group Σ of $AG(2m, 2)$ with $Sp(2m, 2)$. If B is a block, then Γ_B is isomorphic to $Sp(2m, 2)$ and is 2-transitive on B and $\mathscr{C}B$. Moreover, if $x \in B$ then Γ_{xB} is $GO^\epsilon(2m, 2)$.*

Proof. If B, X and Y are distinct then $(B \triangle X) \cap (B \triangle Y) \neq \varnothing$, so the $v - 1$ hyperplanes $B \triangle X$, $X \neq B$, are pairwise non-parallel. We can thus recover $\mathfrak{A}$ from $\mathscr{S}^\epsilon(2m)$, so that Γ is a collineation group of $\mathfrak{A}$ and $\mathfrak{A}^\#$. $\Gamma = \Sigma\Gamma_0 = \Sigma\Gamma_B$. Regarding Γ as a collineation group of $\mathfrak{A}$, we have that Γ_0 preserves an alternating form f', while Γ_{oB}, $o \in B$, preserves a quadratic from Q' in such a way that B is the set of singular points of Q'. Since the blocks of $\mathscr{S}^\epsilon(2m)$ are the sets $B + x$, $x \in G$, the group of the quadratic form Q' permutes them and thus is Γ_{oB}. As o can be any point of B, $|\Gamma_B : \Gamma_{oB}| = k$, and it follows that Γ_o and Γ_B are both isomorphic to $Sp(2m, 2)$. Finally, Γ_{oB} is transitive on $B - \{o\}$, so that Γ_B is 2-transitive on B. Since the same is true for $\mathscr{S}^{-\epsilon}(2m)$, the result follows.

COROLLARY 1. *$Sp(2m, 2)$ acts 2-transitively on the cosets of its subgroups $GO^\epsilon(2m, 2)$.*

COROLLARY 2. *Γ has an involutory outer automorphism θ such that*

(i) *θ centralizes Σ;*

(ii) *θ interchanges Γ_o and Γ_B, where o is a point and B is a block on o; and*

(iii) *If Π is any subgroup of $Sp(2m, 2)$ not fixing (as a whole) any set of $2^{m-1}(2^m \pm 1)$ points of $AG(2m, 2)$, then Π^θ is a second complement to Σ in $\Sigma\Pi$ moving all points.*

Proof. $\theta: x \leftrightarrow B + x$, $x \in G$, is a polarity and hence acts on Γ. θ satisfies (i) and (ii), and thus is an outer automorphism of Γ. Let $\pi \in \Pi$ and $B^\pi = B + z$, $z \in G$. Then, for any x,

$$x^{\theta\pi\theta} = (B + x)^{\pi\theta} = (B^\pi + x^\pi)^\theta$$
$$= (B + z + x^\pi)^\theta = x^\pi + z.$$

Thus, $\Pi^\theta \leqslant \Sigma\Pi$, so θ normalizes $\Sigma\Pi$ by (i). $(\Sigma\Pi)_B = \Pi^\theta$ fixes both B and $\mathscr{C}B$. By hypothesis, Π^θ and Π are not conjugate, which proves (iii).

COROLLARY 3. *$\mathscr{S}^\epsilon(2m)$ can be constructed as follows. Let V be a $2m$-dimensional vector space over $GF(2)$, and Q a nondegenerate quadratic form on V whose group is $GO^\epsilon(2m, 2)$. Let B be the set of singular vectors of Q. Then the points and blocks of $\mathscr{S}^\epsilon(2m)$ are the vectors of V and the translates $B + v$, $v \in V$, respectively.*

It follows that, if Q is as in Corollary 3, B is its set of singular vectors, and $v \in B$, then $B + v$ is the set of singular vectors of a nondegenerate quadratic form associated with the same alternating form as is Q.

THEOREM 2. *Let $\mathscr{D}$ be a symmetric design admitting a sharply point-transitive automorphism group Σ. Define addition of points so that Σ is the set of right translations of the group G of points. Then the following statements are equivalent.*

(i) *$\mathscr{D}$ is $\mathscr{S}^\epsilon(2m)$ for some m and ϵ.*

(ii) *$X \triangle Y$ is a left coset of a subgroup of G whenever X and Y are distinct blocks.*

(iii) *$\mathscr{C}(X \triangle Y)$ is a left coset of a subgroup of G whenever X and Y are distinct blocks.*

Proof. We have already shown that (i) $\Leftrightarrow$ (ii).

(iii) $\Rightarrow$ (ii). We may assume that $k \geqslant 2\lambda$. Suppose $v \neq 4n$. As $(v - 2n) \mid v$, we have $v \geqslant 3(v - 2n)$. Then

$$\lambda(3n - 1) \geqslant \lambda(v - 1) = k(k - 1) \geqslant 2\lambda(k - 1),$$

so $k > 3\lambda - 1$. The same argument now yields $\lambda(3n - 1) \geqslant 3\lambda(k - 1)$, which is impossible. Thus, $v = 4n$, so (ii) is clear.

(ii) $\Rightarrow$ (iii). This follows from (i).

5. The Symmetric Difference Property

In this section we will consider a symmetric design $\mathscr{D}$ satisfying the following condition:

(SDP) *If B, C, D are any three blocks, then $B \triangle C \triangle D$ is either a block or the complement of a block.*

By Lemma 2 of Section 2, $\mathscr{S}^{\epsilon}(2m)$ satisfies (SDP).

THEOREM 3. *Let $\mathscr{D}$ be a symmetric design satisfying* (SDP). *Then the following statements hold:*

(i) *$\mathscr{D}$ has the same parameters as $\mathscr{S}^{\epsilon}(2m)$ for some m and ϵ.*

(ii) *Let $\mathscr{A}$ consist of the points of $\mathscr{D}$ and the symmetric differences of the pairs of blocks of $\mathscr{D}$. Then $\mathscr{A}$ is the set of points and hyperplanes of an affine space $AG(2m, 2)$.*

(iii) *Fix a point o. In $\mathscr{A}$, define addition of points in the natural way, so that o becomes the zero element of a vector space over $GF(2)$. For each point x, let (x) be the set of blocks on x. Then $(o) \triangle (x) \triangle (y) = (x + y)$ or its complement, for all points x, y with $x \neq 0$, $y \neq 0$, $x \neq y$.*

(iv) *The dual of $\mathscr{D}$ satisfies* (SDP).

(v) *Fix a block B. For all X, Y, define the block $X + Y$ by*

$$B \triangle X \triangle Y = X + Y$$

or its complement. Then the blocks form an elementary abelian 2-group under addition.

Proof. (i) and (ii). Suppose that, whenever B, C, D are different blocks, $B \triangle C \triangle D$ is a block. Then $| B \cap C \cap D |$ is independent of B, C, D. By the Dembowski–Wagner Theorem [4], $\mathscr{D}$ is a projective space, and hence fails to satisfy (SDP). Thus, for some B, C, D, $B \triangle C \triangle D$ is not a block, and hence satisfies $B \triangle C \triangle D = \mathscr{C}E$ for a block E. Now $B \triangle C = \mathscr{C}(D \triangle E)$, so $v = 4n$.

Next, for any $B \neq C$ and $D \neq E$, if $B \triangle C \neq D \triangle E$ then

$$|(B \triangle C) \cap (D \triangle E)| = n.$$

For, $D \triangle E = B \triangle X$ or $\mathscr{C}(B \triangle X)$ for some X. It is now elementary to calculate that $|(B \triangle C) \cap (B \triangle X)| = n$, as required.

Call a set $B \triangle C$ or $\mathscr{C}(B \triangle C)$ a hyperplane whenever $B \neq C$. Let $\mathscr{A}$ consist of the points and hyperplanes, and let $\mathscr{A}_x$ consist of the points $\neq x$ and hyperplanes on x.

If H is a hyperplane, each B determines a unique C such that $H = B \triangle C$

or $\mathscr{C}(B \triangle C)$. Thus, counting in two ways the ordered triples (B, C, H) with $H = B \triangle C$ or $\mathscr{C}(B \triangle C)$, we find that there are exactly $2v(v-1)/v = 2(v-1)$ hyperplanes. Consequently, there are an average of $2(v-1) \cdot n/2n = v-1$ hyperplanes per point.

On the other hand, for each point x, the dual of $\mathfrak{A}_x$ has a constant number of blocks per point, and a constant number of blocks per 2 distinct points. Consequently, by a standard incidence matrix argument [3, p. 20], $\mathfrak{A}_x$ has at most $v-1$ hyperplanes. It follows that $\mathfrak{A}_x$ is a symmetric design, and $\mathfrak{A}$ is a 3-design.

Moreover, suppose H and H' are hyperplanes on x. Fix a block B on x. Then $H = B \triangle C$ or $\mathscr{C}(B \triangle C)$, and $H' = B \triangle C'$ or $\mathscr{C}(B \triangle C')$, for some blocks C and C'. Consequently, $x \in \mathscr{C}(H \triangle H') = C \triangle C'$ or $\mathscr{C}(C \triangle C')$, so $\mathscr{C}(H \triangle H')$ is in $\mathfrak{A}_x$. Since $\mathfrak{A}_x$ is a Hadamard design, it is a projective space over $GF(2)$ by the Dembowski–Wagner Theorem [4]. Consequently, $\mathfrak{A}$ is an affine space over $GF(2)$. Now (i) follows from a result of Mann [3, p. 61], since v is a power of 2.

Moreover, the complement of a hyperplane is now also a hyperplane. This proves (ii).

(iii). Let $\{p, q, r, s\}$ be any plane of $\mathfrak{A}$. We must show that $(p) + (q) = (r) + (s)$ or its complement. Let $B, C \in (p) + (q)$; it suffices to show that B and C are both in $(r) + (s)$ or are both in its complement. But B and C are on just one of p, q, so $p, q \in B \triangle C$ or $p, q \in \mathscr{C}(B \triangle C)$. Since $B \triangle C$ is a hyperplane of $\mathfrak{A}$, either $r, s \in B \triangle C$ or $r, s \in \mathscr{C}(B \triangle C)$. Then either B and C are on just one of r, s, or else neither is on just one of these points. Consequently, B and C are both in $(r) + (s)$ or its complement.

(iv). This is clear from (iii).

(v). This is the dual of (iii).

EXAMPLE (N. Patterson). Let f be a nondegenerate alternating bilinear form on a vector space of dimension $2m$ over $GF(2)$. Fix ϵ, and construct $\mathscr{S}^\epsilon(2m)$ as in the proof of Theorem 1. Let W be a totally isotropic subspace of dimension m. Consider the subsets $B \triangle W$ and $\mathscr{C}(B \triangle W)$, where B runs through the blocks of $\mathscr{S}^\epsilon(2m)$. It is not difficult to show that each subset has size $2^{m-1}(2^m \pm \epsilon)$, and that the subsets of size $2^{m-1}(2^m + \epsilon)$ form a symmetric design $\mathscr{D}$ (with point set V). Clearly, $\mathscr{D}$ satisfies (SDP). Further designs satisfying (SDP) can now be obtained from $\mathscr{D}$ and $\mathscr{S}^\epsilon(2k)$ for any k by the tensor product method of Section 3.

An isomorphism between two symmetric designs satisfying (SDP) is, by Theorem 3, induced by a collineation of the underlying affine spaces. It is therefore easy to see that the designs obtained from the above construction are not isomorphic to one another, nor to any $\mathscr{S}^\epsilon(2m)$.

Remark (P. Cameron). Any $(1, -1)$ incidence matrix of a symmetric design $\mathscr{D}$ having $v = 4n$ is a Hadamard matrix H. It is not difficult to check that $\mathscr{D}$ satisfies (SDP) if and only if (i) $v = 2^{2m}$ for some m, and (ii) $H = TH(2m)U$ for some monomial permutation matrices T and U.

6. Further Characterizations

We now present additional characterizations of the designs $\mathscr{S}^{\epsilon}(2m)$, based on other properties of their automorphism groups. We begin with two lemmas.

Lemma 3. *Let γ be a nontrivial automorphism of a symmetric design, where γ has prime order. Then*

(i) *γ fixes at most $\frac{1}{2}v$ points; and*

(ii) *If γ fixes $\frac{1}{2}v$ points, then $v = 4n$ and $\gamma^2 = 1$.*

Proof. (i) is Theorem 3 of Feit [7], while (ii) is essentially contained in the proof of that theorem.

Lemma 4. *Let $\mathscr{D}$ be a symmetric design with $v = 4n$. Let σ and σ' be nontrivial automorphisms, each fixing $2n$ points, such that $F_\sigma \cap F_{\sigma'} = \varnothing$. Then*

(i) *$F_\sigma = \mathscr{C}(B \triangle B^\sigma)$ whenever $B \neq B^\sigma$;*

(ii) *σ and σ' fix no common block;*

(iii) *$F_\sigma = \mathscr{C}(B \triangle C)$ implies that $C = B^\sigma$;*

(iv) *σ is the unique nontrivial automorphism of $\mathscr{D}$ fixing F_σ pointwise; and*

(v) *If $B \neq B^\sigma$ and X is any block, then $|B \cap B^\sigma \cap X| = \frac{1}{2}\lambda$ or $\lambda - \frac{1}{2}n$.*

Proof. (i). σ moves each of the $2n = v - 2n$ points in $B \triangle B^\sigma$.

(ii). Suppose σ and σ' fix a common block. As each moves $2n$ and fixes $2n$ blocks, they must move a common block B. Then $\mathscr{C}(B \triangle B^\sigma) = F_\sigma = \mathscr{C}F_{\sigma'} = B \triangle B^{\sigma'}$, so $\mathscr{C}B^\sigma = B^{\sigma'}$, which is ridiculous.

(iii). $F_\sigma = \mathscr{C}(X \triangle X^\sigma)$ and $\mathscr{C}F_\sigma = F_{\sigma'} = \mathscr{C}(Y \triangle Y^{\sigma'})$ for at least $2n$ blocks X and $2n$ blocks Y. Consequently, the given block B must satisfy (by (ii)) $\mathscr{C}F_\sigma = B \triangle B^\sigma$ or $F_\sigma = \mathscr{C}F_{\sigma'} = B \triangle B^\sigma$. Now $B \triangle C = \mathscr{C}F_\sigma = B \triangle B_o$ or $\mathscr{C}(B \triangle B^\sigma)$ implies that $C = B^\sigma$.

(iv). Let γ be a nontrivial automorphism fixing F_σ pointwise. Then (iii) applies to σ and γ. Consequently, $B^\sigma \neq B$ implies $\mathscr{C}F_\gamma = \mathscr{C}F_\sigma = B \triangle B^\gamma$, so $B^\gamma = B^\sigma$. Similarly, $B^\gamma \neq B$ implies $B^\gamma = B^\sigma$. Thus, $\gamma = \sigma$.

(v). Either $X \triangle X^\sigma = \mathscr{C}F_\sigma$ or $X \triangle X^{\sigma'} = \mathscr{C}F_{\sigma'} = F_\sigma$, so either $n = |X \cap (B \triangle B^\sigma)|$ or $n = |X \cap \mathscr{C}(B \triangle B^\sigma)|$.

THEOREM 4. *Let $\mathscr{D}$ be a symmetric design. Then for some m and ϵ, $\mathscr{D}$ is $\mathscr{S}^\epsilon(2m)$, $PG(m, 2)$, or the complementary design of $PG(m, 2)$, if and only if for any distinct blocks B and C, there is a nontrivial automorphism of $\mathscr{D}$ fixing all points not in $B \triangle C$.*

Proof. $PG(m, 2)$ and its complementary design clearly satisfy the given condition. By Section 4, for $\mathscr{D} = \mathscr{S}^\epsilon(2m)$ each set $\mathscr{C}(B \triangle C)$ is fixed pointwise by a nontrivial elation of the underlying affine space $AG(2m, 2)$.

Conversely, assume that $\mathscr{D}$ satisfies the given condition. Since $\lambda(v-1) = k(k-1)$, we may assume that $v > 2k$. Let T be the given set of automorphisms, where we may assume that each $\sigma \in T$ has prime order. Each $\sigma \in T$ fixes at least $v - 2n$ points. By Lemma 1, $v - 2n \leqslant \frac{1}{2}v$, so $v \leqslant 4n$.

Suppose $\sigma \in T$ moves B. Then the set F_σ of fixed points of σ is disjoint from $B \triangle B^\sigma$, so $\mathscr{C}F_\sigma \subseteq B \triangle B^\sigma$. Since σ fixes at least $v - 2n$ points, $\mathscr{C}F_\sigma = B \triangle B^\sigma$. If $\tau \in T$ also moves B, $\mathscr{C}F_\tau = B \triangle B^\tau$. A simple calculation now proves the following fact.

LEMMA 5. *If σ, $\tau \in T$ move a common block, then either $F_\sigma = F_\tau$ or $|\mathscr{C}F_\sigma \cap \mathscr{C}F_\tau| = n$.*

Assume first that any two elements $\sigma, \tau \in T$ move a common block. Then, by Lemma 5, distinct sets $\mathscr{C}F_\sigma$ have exactly n common points. Let $\mathscr{D}^*$ be the dual of the incidence structure of points and distinct sets $\mathscr{C}F_\sigma$, $\sigma \in T$. Then $\mathscr{D}^*$ has v^* points, $b^* = v$ blocks, $r^* = 2n$ blocks per point, and $\lambda^* = n$ blocks per two distinct points. By a standard incidence matrix argument [3, p. 20], $v = b^* \geqslant v^*$. On the other hand, for a fixed block B there are $v - 1$ sets $B \triangle C$ with $C \neq B$, so $v^* \geqslant v - 1$. Suppose $v^* = v - 1$. Then, for each $\mathscr{C}F_\sigma = X \triangle Y$ and each B, $(X \triangle Y) \triangle B$ is a block. This is impossible by Theorem 3. Consequently, $v^* = v$. Thus, for each $\mathscr{C}F_\sigma$, there is exactly one block B_σ such that $B_\sigma \triangle \mathscr{C}F_\sigma$ is not a block. Let $\mu = |B_\sigma \cap \mathscr{C}F_\sigma|$. Counting in two ways the pairs (x, X) with $x \in X \cap \mathscr{C}F_\sigma$, we find by Lemma 5 that $2n \cdot k = (v-1)n + \mu$. Consequently, $\mu = n(2k - v + 1) \leqslant 0$, so $v - 1 = 2k$ and $\mu = 0$. Then $B_\sigma \subseteq F_\sigma$, so $B_\sigma = F_\sigma$ as both sets have $k = v - 2n$ points. Thus, for any distinct blocks B, C, there are $3 = 1 + (v-1)/k$ blocks containing $B \cap C$. By the Dembowski–Wagner Theorem [4], $\mathscr{D}$ is $PG(m, 2)$ for some m.

We may thus assume that some $\sigma, \sigma' \in T$ move no common block. Then $v \geqslant 2(v - |F_\sigma|) = 4n$, so $v = 4n$. Hence, σ and σ' fix no common block. By the dual of Lemma 4, $F_\sigma \cap F_{\sigma'} = \varnothing$. Let T_0 be the set of $\sigma \in T$ such that some such $\sigma' \in T$ exists, and let $\langle T_0 \rangle$ be the group generated by T_0.

Consider σ and σ'. Take any blocks B and D with $B^\sigma = B$ and $D^\sigma \neq D$. By Lemma 4(ii), $B^{\sigma'} \neq B$, so $B \triangle B^{\sigma'} = \mathscr{C}F_{\sigma'} = F_\sigma = \mathscr{C}(D \triangle D^\sigma)$, and hence $B \triangle D = \mathscr{C}(B^{\sigma'} \triangle D^{\sigma})$. Consequently, if $\tau \in T$ has $F_\tau = \mathscr{C}(B \triangle D)$, then $\tau \in T_0$ and $B^\tau = D$ (by Lemma 4(iii)). It follows that $\langle T_0 \rangle$ is transitive on the blocks and hence the points of $\mathscr{D}$.

We now show that $T_0 = T$. For, suppose $\tau \in T$. The transitivity of $\langle T_0 \rangle$ implies that some $\sigma \in T_0$ must move F_τ. Let σ' be as before. By Lemma 1, $\sigma^2 = \tau^2 = 1$. Then $F_{\sigma\tau\sigma} = F_\tau{}^\sigma \neq F_\tau$, so $\sigma\tau\sigma \neq \tau$. Also, $F_{\sigma\tau\sigma} \neq F_\sigma$ by Lemma 4(iv). Clearly, $F_{\sigma\tau\sigma} \supseteq F_\sigma \cap F_\tau \neq \varnothing$ and $\mathscr{C}F_\sigma \cap \mathscr{C}F_\tau \neq \varnothing$. By Lemma 5, $| F_\sigma \cap F_\tau | = n$. Thus, $F_{\sigma\tau\sigma} \neq F_\sigma, F_\tau$, $| F_{\sigma\tau\sigma} \cap F_\sigma | \geqslant n$, and $| F_{\sigma\tau\sigma} \cap F_\tau | \geqslant n$, so Lemma 5 implies that $F_{\sigma\tau\sigma} \cap F_\sigma = F_{\sigma\tau\sigma} \cap F_\tau = F_\sigma \cap F_\tau$. Then

$$F_{\sigma\tau\sigma} \cap (F_\sigma \triangle F_\tau) \neq \varnothing,$$

so $F_{\sigma\tau\sigma} = \mathscr{C}(F_\sigma \triangle F_\tau)$ as both sets have exactly $2n$ elements. Similarly, $F_{\tau\sigma\tau} \neq F_\sigma$, F_τ (by Lemma 4(iv)), $F_{\tau\sigma\tau} \cap F_\sigma = F_{\sigma\tau\sigma} \cap F_\tau = F_\sigma \cap F_\tau$, and hence $F_{\tau\sigma\tau} = \mathscr{C}(F_\sigma \triangle F_\tau) = F_{\sigma\tau\sigma}$. Again by Lemma 4(iv), $\tau\sigma\tau = \sigma\tau\sigma$. Consequently, $\tau = \sigma\tau\sigma\tau\sigma \in T_0$, as required.

Thus, $T_0 = T$. Let B, C, D be any distinct blocks. Then $B \triangle C = \mathscr{C}F_\sigma = F_{\sigma'}$ for some $\sigma, \sigma' \in T$. By Lemma 4(ii), σ or σ' moves D, so $B \triangle C = D \triangle D^\sigma$ or $\mathscr{C}(D \triangle D^{\sigma'})$. Now Theorem 3 applies: the points of $\mathscr{D}$ and set $B \triangle C$ form an affine space $AG(2m, 2)$. Since F_σ and $F_{\sigma'}$ are parallel hyperplanes fixed pointwise by the nontrivial collineations σ and σ', respectively, $\sigma\sigma'$ must be a translation. Then $\langle T \rangle$ contains the translation group Σ of $AG(2m, 2)$, as F_σ can be any hyperplane. Clearly, Σ_{F_σ} is transitive on F_σ. Consequently, Theorem 4 follows from Theorem 2.

THEOREM 5. *A symmetric design is $\mathscr{S}^\epsilon(2m)$ for some m, ϵ, if and only if its automorphism group is 2-transitive on blocks and contains a nontrivial element fixing at least $\frac{1}{2}v$ points.*

Proof. By Lemma 3, $v = 4n$. If γ fixes $\frac{1}{2}v$ points and moves B, then $\mathscr{C}(B \triangle B^\gamma)$ is its set of fixed points. Since the automorphism group is 2-transitive on blocks [3, p. 79], the result follows from Theorem 4.

LEMMA 6. *Let Π be a transitive collineation group of $PG(d, 2)$ such that the stabilizer of a point x has point-orbits of lengths 1, $2n - 2$, $2n$, where $n = 2^{d-1}$. Let $x^\perp$ consist of x and the points of the orbit of length $2n - 2$. Then $x^\perp$ is a hyperplane and $x \to x^\perp$ is a symplectic polarity preserved by Π.*

Proof. Let Σ be the translation group of $\mathfrak{A} = AG(d + 1, 2)$. $\Gamma = \Sigma\Pi$ is a collineation group of $\mathfrak{A}$. If $x \neq p$ then Γ_{px} has a unique orbit $H(p, x) = H(x, p)$ of length $2n$. Let $q \notin \{p, x\} \cup H(p, x)$. If $\sigma \in \Sigma$ moves p to x, then σ centralizes Γ_{px}. Then $y = q^\sigma \notin H(p, x)$ and $\Gamma_{pqx} = \Gamma_{pqy}$.

As $(2n - 2, 2n) = 2$, Γ_{pqx} has orbits on $H(p, x)$ of lengths divisible by n. In view of the orbit lengths of Γ_{px}, Γ_x is primitive on the points $\neq x$. In particular, $H(p, x) \neq H(p, x')$ if $x \neq x'$. But Γ_{pqx} has two orbits of length n on $H(p, x)$ and also on $H(p, q)$. As $|H(p, q) \cup H(p, x)| < 4n$, it follows that $|H(p, q) \cap H(p, x)| = n$ and $\Gamma_{pqx} = \Gamma_{pqy}$ has exactly 3 orbits of lengths n. $H(p, y)$ must be a union of two of these orbits and is neither $H(p, q)$ nor $H(p, x)$. Thus, $H(p, y) = H(p, q) \triangle H(p, x)$. Similarly, $H(q, y) = H(p, x)$. Since $\{p, q, x, y\}$ is a plane of $\mathfrak{A}$, if $\tau \in \Sigma$ takes p to q it takes x to y and thus fixes $H(p, x)$; this is true for each $q \notin \{p, x\} \cup H(p, x)$. As σ also fixes $H(p, x)$, this set is fixed by $2n - 1$ nontrivial elements of Σ. Then $\Sigma_{H(p,x)}$ is sharply transitive on $H(p, x)$, and $H(p, x)$ is a hyperplane. $z \in x^\perp$ implies that $x \in z^\perp$ (compare [5]), and the lemma follows.

THEOREM 6. *Let $\mathscr{D}$ be a symmetric design admitting a 2-transitive automorphism group Γ such that, for each block B, Γ_B is 2-transitive on both B and $\mathscr{C}B$. If Γ has a regular normal subgroup, then $\mathscr{D}$ is $\mathscr{S}^\epsilon(2m)$ for some m and ϵ.*

Proof. Let Σ be the given normal subgroup. By [13], Lemma 5.5, Σ is an elementary abelian 2-group and $\mathscr{D}$ has parameters (1). We may assume that $\epsilon = -1$.

Let $x \neq p$. By [13], Lemma 4.5, Γ_{px} has two orbits of points $\neq p, x$, having l_i points, y_i of which are not on any block on p and $x (i = 1, 2)$. As in [13], Section 10,

$$l_1 + l_2 = v - 2, \qquad y_1 + y_2 = v - k, \tag{3}$$

$$y_i \leqslant l_i \mid \lambda y_i \qquad (i = 1, 2), \tag{4}$$

$$\lambda y_1^2/l_1 + \lambda y_2^2/l_2 = n^2. \tag{5}$$

We may assume that $4 \nmid l_1$. By (1), (3), and (5),

$$(2^{m-1} - 1)\{y_1^2(v - 2 - l_1) + (v - k - y_1)^2 l_1\} = 2^{3m-3} l_1 l_2 .$$

As $(v - k)^2 \equiv 0 \pmod{2^{2m-2}}$, it follows that

$$(2^{m-1} - 1)\{-2y_1^2 - 2(v - k) y_1 l_1\} \equiv 0 \pmod{2^{2m-2}},$$

or $y_1(y_1 + (v - k) l_1) \equiv 0 \pmod{2^{2m-3}}$. Thus, $y_1 \equiv 0 \pmod{2^{2m-1}}$, since $v - k = 2^{m-1}(2^m + 1)$ and $4 \nmid l_1$. Now (4) implies that

$$y_1 \leqslant l_1 \mid 2(2^{m-1} - 1)(y_1/2^{m-1}) < 2y_1 .$$

Thus, $l_1 = 2(2^{m-1} - 1) y_1/2^{m-1}$. Together with (3) and (5), this yields $l_1 = 2n - 2$, $l_2 = 2n$, $y_1 = 2^{m-1}(2^{m-1} + 1)$, $y_2 = n$. By Lemma 6, the orbit $H(p, x)$ of Γ_{px} of length $2n$ is a hyperplane of $AG(2m, 2)$. Every block on p and x meets $H(p, x)$ in $2n - y_2 = n$ points. Also, there are $\lambda(2n - y_2)/l_2 = \frac{1}{2}\lambda$ blocks on p, x, and a given point of $H(p, x)$. Now an elementary inclusion–

exclusion argument shows that each block on one of the points p, x, meets $H(p, x)$ in λ points, while each block on neither p nor x meets $H(p, x)$ in n points. Consequently, $\Sigma_{H(p,x)}$ fixes the set of blocks on exactly one of the points p, x, so the dual of Theorem 2(ii) holds. This completes the proof.

COROLLARY. *Let V be a d-dimensional vector space over $GF(q)$, and G a subgroup of $\Gamma L(V)$ inducing a rank 3 permutation group on $V - \{0\}$. Form the semidirect product GV, and suppose that there is an intransitive complement to V in GV fixing no vector of V. Then $d = 2m$ is even, q may be assumed to be 2, and $G \leqslant Sp(2m, 2)$.*

Proof. This follows from Theorem 6 and [13], Lemma 4.1 and the proof of Lemma 4.5.

7. THE DESIGNS $\mathscr{D}(\mathscr{O})$

Let $\mathfrak{A}$ be an affine translation plane of order $q = 2^m$ with translation group Σ. The elements of Σ and the points of $\mathfrak{A}$ will be identified. A *line oval* in $\mathfrak{A}$ is a set $\mathscr{O}$ of $q + 1$ lines, one from each parallel class, no three of which concur. In the dual of the projectivization of $\mathfrak{A}$, $\mathscr{O}$ is thus an oval whose knot [3, p. 148] is the dual of the line at infinity. If $\mathfrak{A}$ is desarguesian and this oval is a conic, $\mathscr{O}$ will be called a *line conic.*

Set $B(\mathscr{O}) = \{x \mid x \text{ is on a line of } \mathscr{O}\}$.

THEOREM 7. (i). *$B(\mathscr{O})$ is a difference set in Σ whose corresponding symmetric design $\mathscr{D}(\mathscr{O})$ has parameters* (1) *with $\epsilon = 1$.*

(ii). *If $\mathscr{O}$ is a line conic, then $\mathscr{D}(\mathscr{O})$ admits a 2-transitive automorphism group $\Sigma\Pi$, with Π a collineation group $SL(2, q)$ of $\mathfrak{A}$ fixing a point.*

Proof. (i) Every point of $B(\mathscr{O})$ is on precisely 2 lines of $\mathscr{O}$: $|B(\mathscr{O})| = \frac{1}{2}(q + 1)q$. If $1 \neq \sigma \in \Sigma$, then σ fixes a unique line $L_\sigma \in \mathscr{O}$. Count in two ways the ordered triples (x, L, L') with $x \in L \in \mathscr{O}$ and $x \in L' \in \mathscr{O}^\sigma$:

$$4 \mid B(\mathscr{O}) \cap B(\mathscr{O}^\sigma)| = q(q - 1) + q + q + q.$$

(The terms on the right correspond to the cases $L \neq L_\sigma$, $L \nparallel L' \neq L_\sigma$; $L = L_\sigma, L' \neq L_\sigma$; $L = L_\sigma = L'$; and $L \neq L_\sigma, L' = L_\sigma$.) This proves (i).

(ii). Every collineation of $\mathfrak{A}$ fixing $\mathscr{O}$ induces an automorphism of $\mathscr{D}(\mathscr{O})$. There is a group of such collineations, isomorphic to $SL(2, q)$, which is transitive on both $B(\mathscr{O})$ and $\mathscr{C}B(\mathscr{O})$. The product of this group with Σ is thus a 2-transitive automorphism group of $\mathscr{D}(\mathscr{O})$. Moreover, the stabilizer of a point is $SL(2, q)$.

THEOREM 8. *If $\mathcal{O}$ is a line conic in $AG(2, 2^m)$, then $\mathcal{D}(\mathcal{O})$ is isomorphic to $\mathcal{S}^1(2m)$.*

Proof. Let f be a nondegenerate alternating bilinear form on a 2-dimensional vector space V over $GF(2^m)$. If μ is a nonzero $GF(2)$-linear map from $GF(2^m)$ to $GF(2)$, and V is regarded as a $2m$-dimensional vector space V' over $GF(2)$, then a nondegenerate alternating bilinear form f' on V' is given by

$$f'(v, w) = f(v, w)^\mu, \qquad v, w \in V'.$$

Moreover, every $GF(2^m)$-linear map on V preserving f is also a $GF(2)$-linear map on V' preserving f' (cf. [9, p. 229]). Thus, $\Pi = Sp(2, 2^m) = SL(2, 2^m)$ in its usual representation on V is contained in $Sp(2m, 2)$ in its usual representation on V'. In particular, Π is transitive on $V' - \{0\}$. Let Σ be as in Lemma 1. By Theorem 1, $\Gamma = \Sigma\Pi$ is a 2-transitive automorphism group of $\mathcal{S}^1(2m)$. However, by Theorem 1, Σ is simply the group of translations of the vector space V', hence also the group of translations of V. That is, Σ may be regarded as the group of translations of $AG(2, 2^m)$. It follows that Γ is precisely the group described in Theorem 7(ii). If B is a block of $\mathcal{S}^1(2m)$, then Γ_B is a collineation group of $AG(2, 2^m)$. As Π is transitive on nonzero vectors, Γ_B and Π are not conjugate. Since $\Sigma_B = 1$, Γ_B is isomorphic to Π. There is thus a line conic $\mathcal{O}$ of $AG(2, 2^m)$ such that $B(\mathcal{O})$ is an orbit of Γ_B ([11]; [3, p. 185]). As Γ_B is transitive on B and $\mathcal{C}B$ [3, p. 79], while $|B| = |B(\mathcal{O})|$, it follows that $B = B(\mathcal{O})$. Since the blocks of $\mathcal{S}^1(2m)$ are the images of B under Σ, while the blocks of $\mathcal{D}(\mathcal{O})$ are the images of $B(\mathcal{O})$ under Σ, we have identified $\mathcal{S}^1(2m)$ with $\mathcal{D}(\mathcal{O})$.

THEOREM 9. *Let $\mathfrak{A}$ be a Lüneburg–Tits plane* [12], *so $\mathfrak{A}$ is a translation plane of order $q^2 = 2^{2m}$ with m odd.*

(i). *There is a line oval $\mathcal{O}$ of $\mathfrak{A}$ preserved by a collineation group of $\mathfrak{A}$ isomorphic to the Suzuki group $Sz(q)$.*

(ii). *$\mathcal{D}(\mathcal{O})$ is $\mathcal{S}^1(2m)$.*

Proof. As in the preceding proof, we can regard $Sp(4, 2^m)$ as a subgroup of $Sp(4m, 2)$ transitive on the nonzero vectors. Hence, $\Pi = Sz(q)$ fixes 0 and acts on $\mathcal{S}^1(4m)$. Here, Π has three orbits on vectors, of lengths 1,

$$(q^2 + 1)(q - 1), \qquad (q^2 + 1)\, q(q - 1).$$

Hence, by Corollary 2 of Section 4, Π^θ fixes a point but no block.

On the other hand, $\Sigma\Pi$ is a collineation group of $\mathfrak{A}$, where Σ is the translation group of both $AG(4m, 2)$ and $\mathfrak{A}$. Moreover, $\Pi^\theta < \Sigma\Pi$. By [12] (or [3, p. 186]), it follows that there is a line oval $\mathcal{O}$ preserved by Π^θ. This proves (i), and (ii) is now clear.

We remark that the existence of $\mathcal{O}$ or Π^θ in Theorem 9 was not known in [3] and [12]. Thus, the symmetric designs $\mathcal{S}^1(2m)$ provide a proof of a new property of the Lüneburg–Tits planes. An alternative proof can be given, based on cohomological methods; in fact, there are precisely q conjugacy classes of subgroups of $\Sigma\Pi$ isomorphic to $\Pi = Sz(q)$.

PROBLEM. What geometric conditions on a line oval $\mathcal{O}$ of a translation plane of order 2^m are necessary and sufficient in order that $\mathcal{D}(\mathcal{O})$ be isomorphic to $\mathcal{S}^1(2m)$?

EXAMPLE 1. In $AG(2, 2^m)$, $2^m > 4$, the lines $x = 0$ and $y = mx + m^2$, $m \in GF(2^m)$, form a line oval $\mathcal{O}$. (In fact, in $PG(2, 2^m)$, $\mathcal{O} \cup \{L_\infty\}$ consists of a line conic together with its knot.) However, $\mathcal{D}(\mathcal{O})$ can be shown not to be isomorphic to $\mathcal{S}^1(2m)$. Thus, not every $\mathcal{O}$ determines a design $\mathcal{D}(\mathcal{O})$ isomorphic to a design $\mathcal{S}^1(2m)$.

EXAMPLE 2. Let D be any commutative non-associative division algebra of order 2^m. Then the sets of lines $\{x = 0\} \cup \{y = mx + m^2 \mid m \in D\}$ and $\{x = 0\} \cup \{y = m^2x + m\}$ are line ovals in the translation plane coordinatized by D. It seems very likely that neither type of line oval produces $\mathcal{S}^1(2m)$.

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