Midterm Exam 3 is scheduled for Monday, March 7th, at 3:00pm (in class). Remember that calculators are not allowed for this exam. The following exercises are just a sample of what may be on the exam. Do not restrict your studying to this review, and it is no guarantee that everything on this review will be on the midterm. These exercises are all examples of what knowledge/skills you should have gotten from the material covered so far. Other very important resources for studying include the WebWork assignments, lecture guides and exercises in the text.

These solutions are not complete solutions. I have provided enough information to hint at how to do each problem, but I would expect you to show more work on the exam.

1. The Concept Check (#1-7), True-False Quiz (#1-16,20), and Exercises (#1-14,25-48) in the Chapter 4 Review of the textbook.

   Answers to odd problems in the textbook.

2. Brain weight B as a function of body weight W in fish has been modeled by the power function $B = 0.007W^{2/3}$, where B and W are measured in grams. A model for body weight as a function of body length L (measured in centimeters) is $W = 0.12L^{2.53}$. If, over 10 million years, the average length of a certain species of fish evolved from 15 cm to 20 cm at a constant rate, how fast was this species' brain growing when the average length was 18 cm?

   Let t be time, measured in millions of years. We are given that $\frac{dL}{dt} = \frac{20-15}{10} = 0.5$ cm per million years. The relation $B = 0.007W^{2/3}$ gives $\frac{dB}{dt} = 0.007(2/3)W^{-1/3}\frac{dW}{dt}$, and the relation $W = 0.12L^{2.53}$ gives $\frac{dW}{dt} = 0.12(2.53)L^{1.53}\frac{dL}{dt}$. At the point when $L = 18$ cm, $W = 0.12(18)^{2.53}$ g and $\frac{dW}{dt} = 0.12(2.53)(18)^{1.53}(0.5)$ grams per million years. Then we also have, when $L = 18$ cm, $\frac{dB}{dt} = 0.007(2/3)(0.12(18)^{2.53})^{-1/3}(0.12(2.53)(18)^{2.53}(0.5))$ grams per million years. This means that the fish's brain is growing by $0.007(2/3)(0.12(18)^{2.53})^{-1/3}(0.12(2.53)(18)^{1.53}(0.5))$ grams per million years, when the average length of the fish was 18 cm.

3. A square pyramid is being built so that the length of a side of its base is always equal to the height of the pyramid and the volume contained within the pyramid is growing at $30\,m^3/min$. At what rate is the height growing after an hour? (Note: $V = \frac{1}{3}s^2h$)

   We are given: $\frac{dV}{dt} = 30$ m^3/min. Using the fact that the height is always equal to the length of a side of its base, we have the relation $V = \frac{1}{3}s^3$ and hence $\frac{dV}{dt} = s^2\frac{ds}{dt}$. After one hour (60 minutes), the volume is 1800 m^3 and hence the length of a side of its base is $(5400)^{1/3}$ m. This then gives, after one hour, that $\frac{ds}{dt} = \frac{30}{(5400)^{2/3}}$ meters per minute. This means that, after one hour, the length of a side of the base of this pyramid is increasing at a rate of $\frac{30}{(5400)^{2/3}}$ meters per minute.

4. A 15-foot-long collapsible ladder is sliding down a wall. Beginning 10 feet from the wall, the foot of the ladder is moving away at a steady 2 ft/sec, while the ladder itself is collapsing at 0.5 ft/sec. How fast is the top of the ladder moving initially?

   Let L be the length of the ladder, G be the distance between the wall and the bottom of the ladder, and W be the height of the top of the ladder. We are given that, initially, $L = 15$ ft, $\frac{dL}{dt} = -0.5$ ft/sec, $G = 10$ ft, and $\frac{dG}{dt} = 2$ ft/sec. Using $L^2 = G^2 + W^2$, then at this point, $W = \sqrt{125}$ feet, and we want $\frac{dW}{dt}$. Differentiating, we get the relation $2L\frac{dL}{dt} = 2G\frac{dG}{dt} + 2W\frac{dW}{dt}$. Plugging in the known quantities, we find the the initial rate of change of W is given by $\frac{dW}{dt} = -\frac{27.5}{\sqrt{125}}$ ft/sec. This means that the top of the ladder is initially falling down the wall at a rate of $\frac{27.5}{\sqrt{125}}$ feet per second.

5. Compute each of the following limits:

   (a) $\lim_{x\to 0^+} x\ln(x)$

   This is indeterminate of type $0\cdot\infty$, and so we rewrite it as type $\frac{0}{0}$: $\lim_{x\to 0^+}\frac{\ln(x)}{x^{-1}}$ and use L'Hospital's rule to get $\lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$.

   (b) $\lim_{x\to\infty}\left(x^{-1}\ln(x)\right)$

   This is indeterminate of type $0\cdot\infty$, and so we rewrite it as type $\frac{\infty}{\infty}$: $\lim_{x\to\infty}\frac{\ln(x)}{x}$ and use L'Hospital's Rule to get $\lim_{x\to\infty}\frac{1/x}{1} = 0$.

   (c) $\lim_{x\to 0^+}(1+x)^{1/x}$

   This is indeterminate of type 1^∞, and so we rewrite it by introducing a log: $\lim_{x\to 0^+} e^{\ln((1+x)^{1/x})} = e^{\lim_{x\to 0^+}\frac{\ln(1+x)}{x}}$. The exponent is then indeterminate of type $\frac{0}{0}$ and we use L'Hospital's Rule to get $e^{\lim_{x\to 0^+}\frac{1}{1+x}} = e$.

   (d) $\lim_{x\to 0}\frac{\cos(x)-\cos(2x)}{\cos(3x)-\cos(x)}$

   This is indeterminate of type $\frac{0}{0}$, and so we use L'Hospital's Rule to get $\lim_{x\to 0}\frac{-\sin(x)+2\sin(2x)}{-3\sin(3x)+\sin(x)}$. This is also indeterminate of type $\frac{0}{0}$ and so we use L'Hospital's Rule again to get $\lim_{x\to 0}\frac{-\cos(x)+4\cos(2x)}{-9\cos(3x)+\cos(x)} = -\frac{3}{8}$.

(e) $\lim_{x\to 0} \dfrac{x\ln(1+x)}{e^x - 1 - x}$

This is indeterminate of type $\frac{0}{0}$, and so L'Hospital gives $\lim_{x\to 0} \dfrac{\frac{x}{1+x} + \ln(1+x)}{e^x - 1}$ which is also indeterminate of type $\frac{0}{0}$. Using L'Hospital again and simplifying, we get $\lim_{x\to 0} \dfrac{2+x}{(1+x)^2 e^x} = 2$.

(f) $\lim_{x\to 0} \dfrac{\sin(x) + x^2 - x}{x^3 - x^2}$

This is indeterminate of type $\frac{0}{0}$, and so L'Hospital gives $\lim_{x\to 0} \dfrac{\cos(x) + 2x - 1}{3x62 - 2x}$. This is indeterminate of type $\frac{0}{0}$, and so L'Hospital gives $\lim_{x\to 0} \dfrac{-\sin(x) + 2}{6x - 2} = -1$.

6. The derivative of a function f is given by $f'(x) = x(2x-3)(x+1)^2$. Find each of the following:

(a) any intervals on which f is increasing or decreasing.

Since $f'(x) > 0$ implies that f is increasing, and $f'(x) < 0$ implies that f is decreasing, we determine where f' is positive or negative. This gives us that f is increasing on the intervals $(-\infty, 0)$ and $(3/2, \infty)$, and f is decreasing on the interval $(0, 3/2)$.

(b) any x-values where f has local extrema, and identify each as a maximum or a minimum.

The local extrema can only occur at critical points, which are when $x = 0, -1, 3/2$. Since f changes from increasing to decreasing when $x = 0$, we have a local maximum at $x = 0$. Since f changes from decreasing to icnreasing at $x = 3/2$, we have a local minimum at $x = 3/2$. There is no extrema at $x = -1$.

(c) any intervals on which f is concave up or down.

The second derivative is $f''(x) = (x+1)(x-1)(8x+3)$. $f''(x) > 0$, and hence f is concave up, on the intervals $(-1, -3/8)$ and $(1, \infty)$. $f''(x) < 0$, and hence f is concave down, on the intervals $(-3/8, 1)$ and $(-\infty, -1)$.

(d) any x-values where f has an inflection point.

Inflection points are where f changes concavity. These are when $x = -1, -3/8, 1$.

7. A rancher wants to fence in an area of 60,000 square feet in a rectangular field and then divide it in half with a fence down the middle parallel to one side. What is the shortest length of fence that the rancher can use, to minimize his cost?

Let y be the length of the fence down the middle (in feet) and hence also the length of the two sides parallel to it. Let x be the length of each of the other two sides (again in feet). The length of fence used would then be $2x + 3y$; this is what we want to minimize. Since the area enclosed must be 60,000 square feet, we have $xy = 60000$ and hence $y = 60000/x$. Substituting this in, we are trying to minimize the function $C(x) = 2x + 3 \cdot 60000/x$. Its derivative is $C'(x) = \dfrac{2(x-300)(x+300)}{x^2}$, and hence the cost is minimzed when $x = 300$ feet and $y = 200$ feet.

8. Find each of the following, and then sketch the graph of the function $f(x) = \dfrac{e^x}{x}$.

   (a) Domain: $(-\infty, 0) \cup (0, \infty)$

   (b) Intercepts: None

   (c) Asymptotes: VA at $x = 0$ because $\lim_{x \to 0^+} f(x) = \infty$, and HA at $y = 0$ because $\lim_{x \to -\infty} f(x) = 0$.

   (d) Intervals of Increase/Decrease: increasing on $(1, \infty)$, decreasing on $(-\infty, 0)$ and $(0, 1)$.

   (e) Relative Extrema: Local minimum at $(1, e)$

   (f) Intervals of Concave Up/Down: concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$

   (g) Inflection Points: None

   graph omitted

9. Find all elements of a complete graph, and then sketch the graph, of the function $g(x) = \ln(x^2 + 1)$.

   domain $(-\infty, \infty)$, intercept at $(0, 0)$, no asymptotes, increasing on $(0, \infty)$, decreasing on $(-\infty, 0)$, local minimum at $(0, 0)$, concave up on $(-1, 1)$, concave down on $(-\infty, -1)$ and $(1, \infty)$, inflection points at $(\pm 1, \ln(2))$, graph omitted

10. Find all elements of a complete graph, and then sketch the graph, of the function $h(x) = \dfrac{x-3}{x+2}$.

    domain $(-\infty, -2) \cup (-2, \infty)$, intercepts at $(3, 0)$ and $(0, -3/2)$, vertical asymptote at $x = -2$, horizontal asymptote at $y = 1$, increasing on $(-\infty, -2)$ and $(-2, \infty)$, never decreasing, no local extrema, concave up on $(-\infty, -2)$, concave down on $(-2, \infty)$, no inflection points, graph omitted

11. A plane flying horizontally with a constant speed of 360 km/h passes over a ground radar station at an altitude of 5 km. At what rate is the distance from the plane to the radar station increasing two minutes later?

    Let x be the horizontal distance between the plane and the station, and let D be the actual distance between the plane and the station (both measured in km). This forms a triangle which gives the relation $5^2 + x^2 = D^2$ and hence $2x\dfrac{dx}{dt} = 2D\dfrac{dD}{dt}$. After 2 minutes, $x = 12$ and $\dfrac{dx}{dt} = 360$ and hence $D = 13$ and $\dfrac{dD}{dt} = 12(360)/13$ kilometers per hour. Thus, the distance between the plane and the radar station is increasing at a rate of 12(360)/13 kilometers per hour after two minutes.

12. The radius of a spherical watermelon is growing at a constant rate of 2 centimeters per week. The thickness of the rind is always one tenth of the radius. At what rate is the volume of the rind growing at the end of the fifth week? Assume that the radius is initially zero.

    Let R be the radius of the watermelon, and let r be the thickness of the rind. Then the volume of the rind is the volume of watermelon minus the volume of the inside, hence $V = \frac{4}{3}\pi R^3 - \frac{4}{3}\pi(R - r)^3$. Since $r = \frac{1}{10}R$, we substitute and get $V = \frac{243}{250}\pi R^3$ and hence $\dfrac{dV}{dt} = \frac{729}{250}\pi R^2 \dfrac{dR}{dt}$. After 5 weeks, $R = 10$ and $\dfrac{dR}{dt} = 2$ and so $\dfrac{dV}{dt} = 216.8$ cubic centimeters per week.

13. The manager of an apartment complex knows from experience that 100 units will be occupied if the rent is set at \$600 per month. A market survey suggests that, on the average, one additional unit will remain vacant for each \$5 increase in monthly rent. Similarly, one additional unit will be occupied for each \$5 decrease in monthly rent. What rent should the manager charge to maximize revenue? (Make sure to include units in your answer)

    Let x be the monthly rent. Then the number of units that will be occupied is given by $-\frac{1}{5}x + 220$, hence the revenue is $R(x) = -\frac{1}{5}x^2 + 220x$. Since $R'(x) = -\frac{2}{5}x + 220$, the revenue is maximized when the monthly rent is \$500.