Final Review: Mathematics 432/532

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1 Introduction

The final will cover 432/532. But for fun, let's look back over the entire course. This first section is not designed to help you study for the final, but instead to help you place our course in the general context of modern mathematics.

- Spaces can be constructed by gluing simple things together. We constructed projective spaces, Klein bottles, and lens spaces that way. The simple pieces live in R^n, but the new objects usually don't. (To be more precise, it is irrelevant and distracting to find an embedding of the new objects into R^n.) We introduced topological spaces and particularly the quotient topology so we could talk about the glued objects without worrying about embedding them in R^n.

- Spaces can be shown homeomorphic by cutting them into pieces and gluing them together in a different way. Using this technique, we proved that $T^2 \# RP^2$ is homeomorphic to $K \# RP^2$, we classified compact surfaces, and we proved that $L(7,2)$ and $L(7,4)$ are homeomorphic.

- If two spaces are homeomorphic (in this way or another way) they must have the same fundamental group. We computed the abelianized fundamental groups $H_1(\mathcal{S})$ for surfaces. The abelianized fundamental group of $T^2 \# T^2 \# \ldots \# T^2$ is

  $$Z \times \ldots \times Z$$

  The abelianized fundamental group of $RP^2 \# RP^2 \# \ldots \# RP^2$ is

  $$Z \times \ldots \times Z \times Z_2$$

  In particular, no two surfaces in our canonical list of compact surfaces can be homeomorphic. Thus the fundamental group puts limits on what can be accomplished by cutting an object into pieces and reassembling it.

- We computed the fundamental groups of lens spaces, and thereby proved that $L(p, q)$ cannot be homeomorphic to $L(\tilde{p}, \tilde{q})$ unless $p = \tilde{p}$. We computed the fundamental groups of the complements of knots, and proved that the trefoil knot is not trivial.

- We have two methods of calculating fundamental groups. The first uses the theorem of Seifert-Van Kampen and computes from a knowledge of the fundamental groups of $\mathcal{U}, \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V}$. To compute these easier groups, we often use strong deformation retracts to reduce to the fundamental groups of simplier objects. In the end, these methods require a knowledge of $\pi(S^1)$.

- The second method relies on covering spaces. We can find the fundamental group of X if we can construct the universal cover $\tilde{X}$ and then compute the deck transformation group Γ; indeed the fundamental group equals this deck transformation group. For example, the universal cover of S^1 is R and the deck transformation group is generated by $x \rightarrow x + 1$ and thus Γ and $\pi(S^1)$ are isomorphic to Z. This technique allows us to compute the fundamental groups of circles, tori, cylinders, Klein bottles, Mobius bands, and projective spaces. It does not work well for other surfaces because their universal covering spaces are homeomorphic to the non-Euclidean plane and their deck transformation groups are groups of non-Euclidean motions.

- We can classify covering spaces of a given X by finding subgroups of the fundamental group.

- Indeed we can get our hands on X if we only know $\tilde{X}$ and a group of transformations which act on $\tilde{X}$ without fixed points; this group of transformations plays the role of the deck transformation group Γ and so $X = \tilde{X}/\Gamma$.

- For example, we provided an alternative description of lens spaces this way starting with an action of Z_p on S^3 in which nonzero elements of the group act without fixed points.

- We classified surfaces by cutting objects into pieces and gluing in a different way. This technique also works for lens spaces, but nobody has succeeded in making it work in general for compact 3-manifolds. However, the alternate covering space techniques from this term do lead to general classification results in 3 dimensions, modulo a couple of unproved conjectures.

- For instance, it is conjectured that all compact three manifolds X with finite fundamental group arise from groups acting on S^3 exactly as we obtained the lens spaces. A complete list of such manifolds is known. The list arises because it is easy to find finite subgroups of $SO(4)$ which act without fixed points, Indeed $SO(4)$ is essentially $SO(3) \times SO(3)$, as we proved in this course, and finite subgroups of $SO(3)$ are known to be cyclic, dihedral, or the symmetry groups of the tetrahedron, cube, or icosahedron.

But to finish the argument, it is necessary to show that the universal cover of X is S^3. This was conjectured by Poincare long ago in the form *every compact simply connected 3-manifold is homeomorphic to* S^3. This conjecture may have been proved by Perelman; the proof is being reviewed as we speak. It is also necessary to show that the homeomorphism from $\tilde{X}$ to S^3 can be chosen so that each deck transformation becomes a linear transformation on $S^3 \subseteq R^4$. Perelman claims to have proved this as well.

2 Review of the First Half of the Course

I recommend reading the first review sheet for details, but here are the important matters from the first term of our course. Several of these results were actually discussed after the first midterm.

- We introduced the notion of homotopy between two maps $f, g : X \to Y$. Using this notion, we defined the fundamental group $\pi(X, x_0)$.
- We computed $\pi(S^1)$ using covering spaces techniques. This argument is summarized in the first question from the sixth assignment. Roughly speaking, a closed path in S^1 starting and ending at 1 can be lifted to a path in R starting at the origin. This lifted path ends at some integer, which completely determines the homotopy class of the original path in S^1. So $\pi(S^1) = Z$.
- Generalizing this argument, we defined covering spaces $\pi : \tilde{X} \to X$. Each point $p \in X$ must have an evenly covered open neighborhood $\mathcal{U}$, so $\pi^{-1}(\mathcal{U}) = \cup\, \mathcal{U}_\alpha$ where the $\mathcal{U}_\alpha$ are disjoint open sets homeomorphic via π to $\mathcal{U}$.
- Next we studied lifting properties. The fundamental diagram is listed below. Our **First Main Covering Space Theorem** says that $\tilde{f}$ is unique if it exists at all, provided Y is connected. (I might ask for a proof.)

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}_0) \arrow[d, "\pi"] \\
(Y, y_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (X, x_0)
\end{tikzcd}$$

- Our **Second Main Covering Space Theorem** (introduced gradually over several weeks) guarantees the existence of $\tilde{f}$ under increasingly general conditions:
  - $Y = [0, 1]$
  - $Y = [0, 1] \times [0, 1]$
  - Y is simply connected and locally pathwise connected
  - $f_\star(\pi(Y)) \subseteq \pi_\star(\pi(\tilde{X}))$ and Y is locally pathwise connected

  I might ask for a sketch of the proof of any of the first three results, but won't ask about the last one.
- A corollary of 3) is that a universal covering space of X is unique if it exists at all. (To prove this, we must assume that X is locally pathwise connected, which we assume from now on.)
- A corollary of 1) and 2) is that if $\tilde{X}$ is a universal covering space, then $\pi(X, x_0)$ is in one-to-one correspondence with $\pi^{-1}(x_0)$.
- A corollary of 3) is that if $\tilde{X}$ is a universal covering space, then the deck transformation group Γ acts transitively on $\pi^{-1}(x_0)$.
- Putting the two previous items together, we fix $\tilde{x}_0 \in \pi^{-1}(x_0)$ and then there is a one-to-one correspondence between Γ and $\pi(X, x_0)$. This correspondence is actually a group isomorphism.
- The **Third Main Covering Space Theorem** asserts that X has a universal cover $\tilde{X}$, provided X is locally pathwise connected and semi-locally simply connected. You should know this result, but I won't ask for a proof.
- Finally, we used part three of the second main theorem, and the third main theorem, to completely classify covering spaces of a locally pathwise connected, semi-locally simply connected X. I will not ask about these results. The final outcome is that there is a one-to-one correspondence between isomorphism classes of covering spaces of X and conjugacy classes of subgroups of $\pi(X)$.

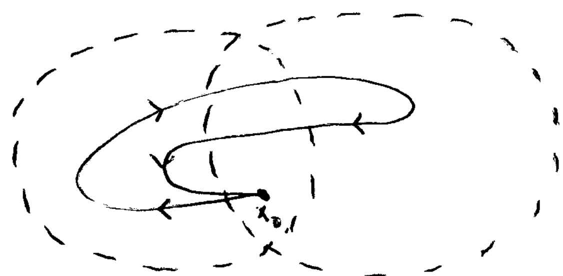
3 Seifert and Van Kampen

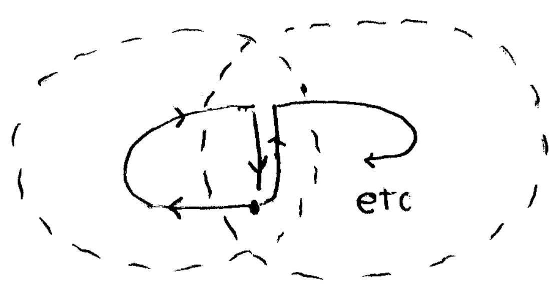
Theorem 1 *Suppose $\mathcal{U}, \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V}$ are pathwise connected open subsets of X and $X = \mathcal{U} \cup \mathcal{V}$. Then $\pi(X)$ is determined by the following diagram.*

$$\begin{tikzcd}
 & \pi(\mathcal{U}) \arrow[dr] & \\
\pi(\mathcal{U} \cap \mathcal{V}) \arrow[ur] \arrow[dr] & & \pi(X) \\
 & \pi(\mathcal{V}) \arrow[ur] &
\end{tikzcd}$$

In more detail, $\pi(X)$ is generated by $\pi(\mathcal{U})$ and $\pi(V)$, and all relations in $\pi(X)$ between elements of these groups is a consequence of starting with an element of $\pi(\mathcal{U} \cap \mathcal{V})$ and identifying the images of this element in $\pi(\mathcal{U})$ and $\pi(\mathcal{V})$.

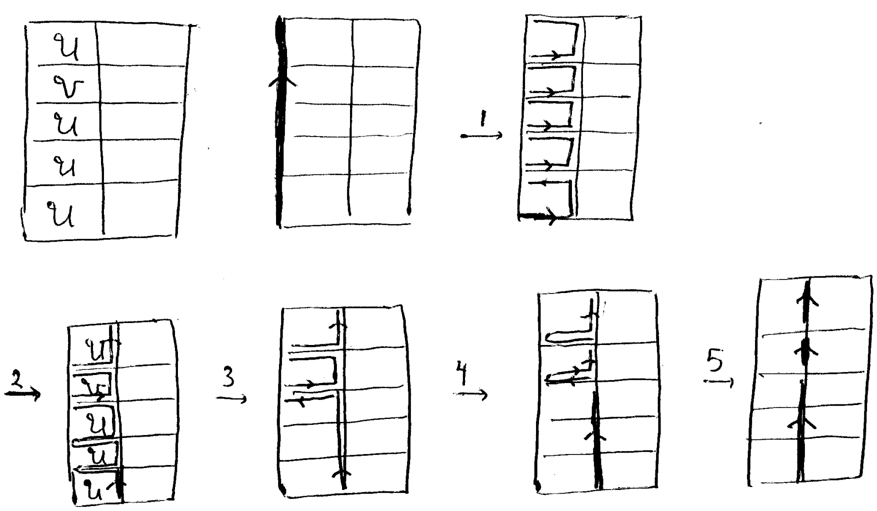
Remark: You should be able to sketch the proof of this result. The proof falls into two pieces. First we prove that every element of $\pi(X)$ is a product of elements in $\pi(\mathcal{U})$ and $\pi(\mathcal{V})$. The picture below illustrates the proof.





The second part of the proof is harder. Consider an example first. Suppose $\pi(\mathcal{U}) = F(a)$ and $\pi(\mathcal{V}) = F(b)$ and $\pi(\mathcal{U} \cap \mathcal{V})$ induces a relation $a^3 = b^2$. A consequence is that the words $a^5 \star b^{-1} \star a$ and $a^2 \star b \star a$ are equal. This might be proved as follows: $(a^5) \star b^{-1} \star a = (a^2 \cdot a^3) \star b^{-1} \star a = a^2 \star (b^2 \cdot b^{-1}) \star a = a^2 \star b \star a$. In this argument, the key step is the second equality, which replaces $a^3 \in \pi(\mathcal{U})$ by $b^2 \in \pi(\mathcal{V})$.

In the general proof, we suppose that we have a path γ expressed as a product of elements in $\pi(\mathcal{U})$ and other elements in $\pi(\mathcal{V})$, and a second such path τ, and a homotopy h between these paths. By standard arguments, we can subdivide $I \times I$ such that each subsquare is mapped to $\mathcal{U}$ or $\mathcal{V}$. We connect the vertices of this subdivision to x_0 by paths. Consequently, each horizontal or vertical segment in the grid is itself in $\pi(\mathcal{U})$ or $\pi(\mathcal{V})$.



We then move γ on the left to τ on the right by a series of moves, paying close attention to how these moves affect the words which spell γ on the left and τ on the right. We move over one column at a time. In the picture below, step #1 just replaces elements in $\pi(\mathcal{U})$ by products of other elements in $\pi(\mathcal{U})$ (and ditto for $\pi(\mathcal{V})$). In steps #2 and #3, we ignore the horizontal segments at the top and bottom because the homotopy is constantly x_0 at these points. We also cancel horizontal segments and their inverses within $\mathcal{U}$.

In the crucial step #4, we replace a horizontal element moving left in $\mathcal{U}$ with a corresponding horizontal element moving left in $\mathcal{V}$, using the relations from Seifert-Van Kampen. This allows both the left moving horizontal element and its right moving inverse to belong to the same $\pi(\mathcal{V})$ where they cancel. In step #5 this cancellation is performed and we have moved over one column. Continue.

4 Consequences

Many fundamental groups can be computed using this theorem. Here is a list of examples. You should know how these calculations are performed.

- $\pi(S^n)$ for $n \geq 2$
- $\pi(RP^n)$ for $n \geq 2$
- $\pi(B)$ where B is a bouquet of circles
- $\pi(\mathcal{S})$ for any surface. For example, the two-holed torus with fundamental polygon $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$ has fundamental group with generators a_1, b_1, a_2, b_2 and one relation obtained by setting the above symbol equal to e.
- $\pi(L(p,q))$
- $\pi(RP^n \# RP^n)$ where # represents a connected sum. There is a difference between the cases $n = 2$ and $n \geq 3$.
- $\pi(R^3 - K)$ where K is a tame knot. In this case, you should know how to calculate this group by producing generators and relations, but you need not know how to use Seifert-Van Kampen to justify the calculation.

5 $H_1(X)$

By definition, $H_1(X) = \pi(X)/[\pi(X), \pi(X)]$. That is, it is the fundamental group, but made abelian by introducing additional relations which guarantee that all elements commute. We proved the following results, which you should be able to reconstruct:

- $H_1(X) = Z \times \ldots \times Z$ if X is a torus with g holes, where there are $2g$ copies of Z
- $H_1(X) = Z \times \ldots \times Z \times Z_2$ if $X = RP^2 \# \ldots \# RP^2$, where there are g copies of RP^2 and $g - 1$ copies of Z
- $H_1(X) = Z$ if X is the complement of a knot
- $H_1(X) = \pi(X)$ if X is a topological group

In particular, $H_1(X)$ is a complete invariant which distinguishes two compact surfaces whenever they are not homeomorphic. On the other hand, $H_1(X)$, for X the complement of a knot, is always the same, so $H_1(X)$ is useless if we want to distinguish knots.

6 Lens Spaces

We studied lens spaces in the last two weeks of the term. These spaces are extremely useful examples of three dimensional manifolds. You should understand our two definitions: gluing the boundary of a ball and forming a quotient space of S^3 under the action of Z_p, and why these two definitions give the same space. You should be able to explain why the lens space is a compact 3-dimensional manifold. Finally, you should be able to compute its fundamental group in two different ways.

In particular, RP^3 is the lens space $L(2,1)$. Why? The fundamental group of this space is Z_2. Draw a generator of this group, and explain why the square of the generator is trivial, but the generator is not trivial.

The fundamental group of $L(p,q)$ is Z_p. Draw a generator of this fundamental group. Explain why the sum of p copies of this generator is trivial.

7 Sample Exercises

The following exercises are interesting and could be profitably studied for the final.

- Sixth assignment: Review the covering space arguments which allow us to compute $\pi(S^1)$. Then classify covering spaces of the Mobius band. Explain why the torus has two 2-fold coverings which are homeomorphic as topological spaces but not isomorphic as covering spaces. Find several covering spaces of a bouquet of two circles, and find the corresponding subgroups of $F(a,b)$.
- Seventh assignment: Exercises 22.3a, 22.3c, 23.1b and extra problems 1 and 2.
- Eighth assignment: Exercise 3 showing that $L(p,q)$ is a manifold, and Exercise 4 computing its fundamental group.
- Nineth assignment: 24.4b, 24.4c, 25.1b, 25.1f-h on computing fundamental groups using Seifert-Van Kampen. The very last exercise showing that the covering space definition of a lens space is the same as the cut and paste definition of a lens space.