Assignment 2. Due Friday, April 8.

Read pages 12 - 33 of the notes. Pages 25 - 33 begin surface theory; it is sufficient to skim these for now.

1. In problem five of assignment 1, you discovered a formula for the curvature of $\gamma(t)$. Several formulas are possible. Show that

$$\kappa = \frac{||\gamma' \times \gamma''||}{||\gamma'||^3}$$

2. Suppose $y = f(x)$ is a curve in the plane. Prove the following formula, which was obtained by Newton in 1673:

$$\kappa = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}$$

3. Recall that N is not defined if $\kappa = 0$. Consider the curve $\gamma(t) = (t, t^3, t)$. Compute $\kappa(t)$ and $N(t)$. Notice that N is not defined at $t = 0$. Show that it is impossible to extend the definition of N to $t = 0$ to make $N(t)$ continuous.

4. In the plane, there is an alternate way to define N for all t. Let N be the vector T rotated counterclockwise by 90 degrees. The equation $\frac{dT}{du} = \kappa N$ then holds provided we allow κ to be negative. Consider the special case of the sine curve $y = \sin(x)$ traced from left to right and using this definition of N. Draw a picture of the curve and at various points along the curve draw T and N. Guess what the graph of the function $\kappa(x)$ will look like and draw a sketch of this graph. Then compute κ and N and verify that your guess was correct. What are the center and radius of the osculating circle at $x = \frac{\pi}{2}$?

5. Return to the curve $\gamma(t) = (t, t^3, t)$. Show that it is possible to define unit normal vectors $N(t)$ continuously for all t such that your normal is plus or minus the normal obtained in problem three whenever $t \neq 0$ is a particular time. Compute κ such that $\frac{dT}{du} = \kappa N$ and compare it with the κ obtained in problem three.

6. Let κ and τ be constants. Suppose $\gamma(u), T(u), N(u)$, and $B(u)$ are obtained by solving the Frenet-Serret formulas for κ and τ with initial conditions $\gamma(0)$, $T(0), N(0)$, and $B(0)$ where these last three vectors are perpendicular unit vectors. If $\kappa = 0$, show that the solution is a straight line. In this case, explain how T, N, and B behave (their behavior will depend on τ).

7. In the previous exercise, suppose $\kappa \neq 0$ and $\tau = 0$. Show that γ is a circle.

8. In the previous exercise, suppose $\kappa \neq 0$ and $\tau \neq 0$. Show that γ is a helix.

9. From page 79 of the book, do exercises 6.6 and 6.8. From pages 91 and 92 do exercises 7.3, 7.5, 7.6, 7.7.