Assignment 3. Due Friday, April 15.

There is a common theme in this assignment. Suppose $s(u, v)$ is a parameterized surface. Then $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$ evaluated at a particular point generate the tangent plane to the surface at that point. So every tangent vector on the surface has the form

$$A\frac{\partial s}{\partial u} + B\frac{\partial s}{\partial v}$$

for constants A and B.

The vector $n = \frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v}$ is normal to the surface at the particular point p_0 in question and therefore the equation of the plane tangent to the surface at p_0 is

$$\{\ p \mid (p - p_0) \cdot n = 0\ \}.$$

If $\gamma(t) = (u(t), v(t))$ is a path in local coordinates, then the corresponding path on the surface is $s(u(t), v(t))$. The derivative of this path on the surface is

$$\frac{\partial s}{\partial u}\frac{du}{dt} + \frac{\partial s}{\partial v}\frac{dv}{dt}.$$

This derivative is indeed a linear combination of $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$, so it is tangent to the surface as expected. The corresponding derivative of the path in local coordinates is $\left(\frac{du}{dt}, \frac{dv}{dt}\right)$. Therefore the general rule for transforming vectors (A, B) in the coordinate plane to vectors tangent to the surface is

$$(A, B) \to A\frac{\partial s}{\partial u} + B\frac{\partial s}{\partial v}$$

Last but not least, if (A_1, A_2) and (B_1, B_2) are vectors in local coordinates, then we do not measure their inner produce by the standard inner product $A_1B_1 + A_2B_2$. Instead we measure their inner product by transferring the vectors to tangent vectors to the surface in R^3 and then computing the inner product of these three dimensional vectors. This gives an inner product

$$< A, B >= \sum g_{ij}A_iB_j$$

in the coordinate plane, where

$$g_{ij} = \frac{\partial s}{\partial u_i} \cdot \frac{\partial s}{\partial u_j}$$

and this dot product is the usual dot product in three space.

As one special case, if $\gamma(t) = (u_1(t), u_2(t))$ is a curve in the coordinate plane, then by the length of this curve we really mean the length of the corresponding curve on the surface in three space, and this length is given by

$$\int \sqrt{\sum g_{ij} \frac{du_i}{dt} \frac{du_j}{dt}} \, dt$$

which generalizes the standard formula below, valid in the special case $g_{ij} = \delta_{ij}$:

$$\int \sqrt{\left(\frac{du_1}{dt}\right)^2 + \left(\frac{du_2}{dt}\right)^2} \, dt$$

This assignment contains some integrals which cannot be done in closed form. Feel free to use any technique you like to approximate them. I like Mathematica, which is available in several labs and also on gladstone and darkwing. To run Mathematica while logged in from a Terminal, type "math" and press enter. Enter "Quit" to quit.

Numerical integrals can be found using the following syntax. Notice that functions are always capitolized and arguments are inside square brackets. Also notice that a space indicates multiplication.

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Integrate[x^2, {x, 0, 1}]

Answer: 1/3

Integrate[Sqrt[1 + 4 t^2], {t, 0, 1}]

Answer: (2 Sqrt[5] + ArcSinh[2]) / 4

Integrate[Sqrt[1 + 4 t^2], {t, 0, 1}]//N

Answer: 1.47894  (so the extra //N requests a decimal value)

Integrate[Sqrt[1 + Sin[x]^2 Cos[x]^2], {x, 0, 1}]

Answer: EllipticE[2, - 1/4] / 2

Integrate[Sqrt[1 + Sin[x]^2 Cos[x]^2], {x, 0, 1}]//N

Answer: 1.0709
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1. Consider the surface $z = x^2 + y^2$. Draw a picture of the surface. A parametric form of the surface is $s(u, v) = \left(u, v, u^2 + v^2\right)$. Find a normal vector to the surface at the point $(1, 2, 5)$. Your normal need not have unit length. Then find the equation of the plane tangent to the surface at this point.

2. Show that $(2, 7, 27)$ is on this tangent plane. Since $(1, 2, 5)$ is also on the plane, the difference $(1, 5, 22)$ is a vector tangent to the surface. Show that this vector is indeed a linear combination of $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$. Using this combination, find the corresponding tangent vector in the uv-plane.

3. Actually, whenever a surface is given by $z = f(x, y)$ it is easy to start with a tangent vector to the surface and find the corresponding vector in local coordinates. Using the previous exercise, you should be able to guess the rule. Show that it always holds.

4. But sometimes the situation is more complicated. Let $s(u, v) = (u^2, u + v, uv)$. near $u = v = 1$. This point maps to $(1, 2, 1)$. Find the equation of the tangent plane to the surface at this point. Show that $(3, 5, 4)$ is on this tangent plane. Hence the vector from $(1, 2, 1)$ to $(3, 5, 4)$ is tangent to the surface. Show that it is indeed a linear combination of $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$. Using this combination, find the corresponding tangent vector in the uv-plane.

5. Return to the surface $z = x^2 + y^2$. Compute $g_{11}, g_{12} = g_{21}$, and g_{22}.

6. Using the previous exercise, find the length of the vector $(1, 5)$ at $(1, 2)$ using the fancy inner product $< A, B >$. Show that your result is indeed the length of the corresponding tangent vector to the surface in three space.

7. Find the cosine of the angle between the vector$(1, 5)$ and the vector $(2, 2)$ at $(1, 1)$ using the fancy inner product. Show that your result is indeed the cosine of the angle between the corresponding tangent vectors in three space.

8. Compute the length of the circle $\gamma(t) = (\cos t, \sin t)$ using the fancy inner product. Explain why your result gives the length of the corresponding circle on the surface $z = x^2 + y^2$. This explanation can be a very simple picture.

9. Now compute the radius of the above circle using the fancy inner product. There are lots of radii; pick a particularly easy one to do the computation. You will discover that the length of the circle is *not* 2π(radius), but is instead smaller. Explain why.

10. From now on, consider the saddle $z = x^2 - y^2$. Parameterize this as $s(u, v) = (u, v, u^2 - v^2)$. Compute $g_{11}, g_{12} = g_{21}$, and g_{22}. Then compute the length of the circle $\gamma(t) = (\cos t, \sin t)$. Actually, you'll get an integral which is impossible to integrate. Try to get a rough approximate value. Show that this value is larger than the usual value of a circle of radius one. By drawing a picture of the surface, explain why you'd expect this.

11. However, if this really a circle on the saddle? Find the length of the radius $\gamma(t) = (t, 0)$ for $0 \leq t \leq 1$. Find the length of the radius $\gamma(t) = (t, t)$ for $0 \leq t \leq \frac{1}{\sqrt{2}}$. It suffices to find numerical values for these integrals. Comment on your results.