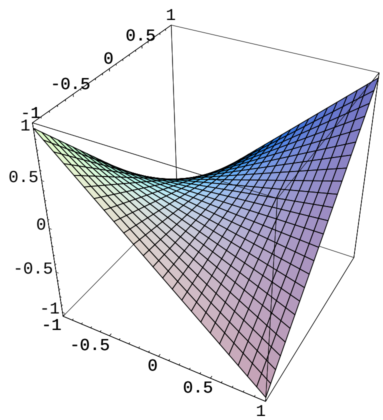
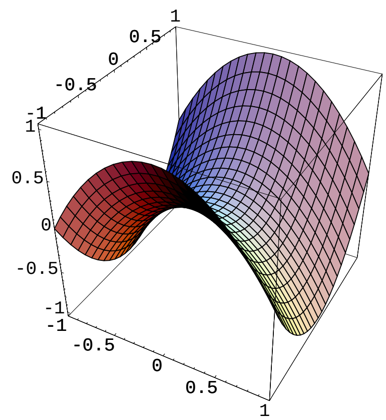
Assignment 4. Due Friday, April 22.

1. The surface $z = xy$ is just a saddle. A picture is shown on the left below. Believe it or not, the picture on the right is the same surface rotated about the z-axis by 45 degrees.





   Compute g_{11}, g_{12}, and g_{22}. In class we usually write these as functions of u and v; feel free to write them as functions of x and y if you prefer. Then consider the curve $\gamma(t) = (t, 0)$ for $0 \le t \le 2$. Compute the length of this curve using the formula $\int \sqrt{< \gamma', \gamma' >}\, dt$. The length of curves on a surface is usually hard to compute, but the length of this curve is a very simple number. Mark the curve γ lifted to the surface on the left, and explain why it has such a simple length.

2. Repeat the previous exercise for the curve $\gamma(t) = (1, 1 - t)$ for $-1 \le t \le 1$. Again, explain from the picture why the length is so simple.

3. Repeat the previous exercise for the curve $\gamma(t) = (t, 1 - t)$ and $-1 \le t \le 1$. This time the length is not simple. Why not?

4. We continue to study $z = xy$. Attach a frame to each point on the coordinate plane by setting $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Compute the lengths of the two vectors; your answer will depend on x and y. Also compute the angle between the two vectors; this angle will depend on x and y. In particular, compute the angle between the vectors at $(1, 1)$ and $(-1, 1)$.

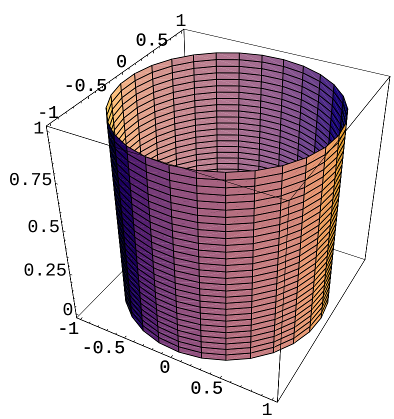
5. Suppose we go out alone the line $y = x$. Show that the angle between e_1 and e_2 is 90 degrees at the origin, and gradually approaches zero as we proceed out. If we go out along the line $y = -x$, show that the angle is 90 degrees at the origin and gradually approaches 180 as we proceed. Explain this result from the picture.

6. In the previous situation, find orthonormal vectors X and Y at each point of the coordinate plane by applying the Gram-Schmidt process. Start by letting $X = \frac{e_1}{||e_1||}$; compute this vector in terms of x and y. Then find a unit vector Y perpendicular to X. Draw a rough sketch of your frame field.

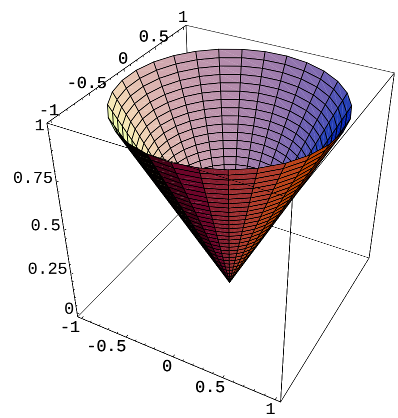
7. There are lots of straight lines on the saddle. Show that every point (x, y, z) on the saddle $z = xy$ is on two straight lines in R^3 which lie entirely in the saddle.

8. Consider the trivial surface $z = 0$ and parameterize using polar coordinates $s(r, \theta) = (r \cos\theta, r \sin\theta, 0)$. Here $r > 0$ and $-\pi < \theta < \pi$. Compute g_{ij}. Recall that geodesics are curves which are locally the shortest distance between their points. The geodesics on the trivial surface $z = 0$ are straight lines. Find the equations of these lines in polar coordinates, and draw a few of them. In particular, notice that geodesics are *not* usually straight lines in the $r\theta$-plane. (Don't do this by solving the geodesic differential equation; just determine what the equation of a staight line in polar coordinates looks like.)

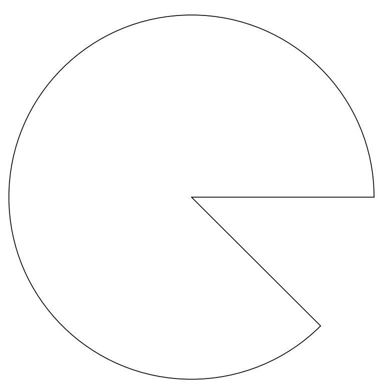
9. Consider the cylinder below. Parameterize via $s(\theta, h) = (\cos\theta, \sin\theta, h)$. Compute g_{ij}. Show that the resulting geometry on the coordinate plane is just Euclidean geometry and thus the geodesics are known. Using this result, describe completely all geodesics on the cylinder.

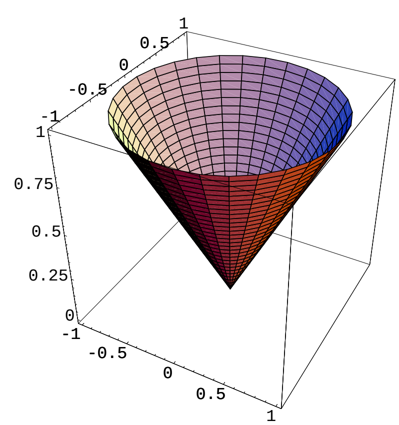


10. Consider the surface $z = \sqrt{x^2 + y^2}$. This surface is a cone. Parameterize via $s(r, \theta) = (r\cos\theta, r\sin\theta, r)$. Compute g_{ij}. Compare with problem 8. Find the equations of geodesics in the coordinate plane.



11. There is, however, a much more illuminating coordinate system. Did you make paper cones in kindergarten? If so, here is a picture of what you did. From this picture, write down a parameterization of the cone $z = \sqrt{x^2 + y^2}$ for which the g_{ij} produce Euclidean geometry; i.e., $g_{ij} = \delta_{ij}$. What are the geodesics in the coordinate plane?





12. Using the previous result, describe all geodesics on the cone. For instance, what geodesics start at the origin? Can a geodesic start at the origin and spiral outward as on the left? Can a geodesic wrap completely around the cone as on the right? What do geodesics look like as they approach infinity?