Assignment 5. Due Friday, April 29.

Midterm: Friday, April 29. This midterm will cover chapter one on curves, sections 1.1 through 1.8. You should also know the formula for curvature when γ is not parameterized by arc length, given on page 20. The exam will not cover the corresponding formula for τ in this case.

The examination will also cover chapter 2, sections 2.1 and 2.2, and 2.6 through 2.12. I'll limit the amount of material from 2.12 as we discuss it in class. Notice that you need not know the material on directional derivatives, vector fields, and the Lie bracket.

I'll give you a review sheet early next week.

Let's keep this week's homework short. But I want you to compute some Christoffel symbols so you'll feel confident with them. Please turn in the first seven exercises on Friday; these will prove useful on the midterm. The remaining two exercises are due a week from Friday; I'll give some additional exercises for that assignment. Since the last two exercises are related to the remaining exercises, I'm putting them on this sheet.

1. Consider the unit disk and define g_{ij} on the interior by the formula

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}$$

   Here I'm using the classical symbolism and you need to make the (easy) translation to g_{ij} notation.

   This example gives non-Euclidean geometry, and many mathematicians have spent their entire lives unraveling the consequences. Hilbert proved that *no surface in* R^3 will give these g_{ij}, while Nash proved that some surface in high dimensions does give it.

   Consider the straight line along the x-axis from the origin to the point $(r, 0)$ where r is some number less than one. Compute the length of this curve. Your formula will give distance as a function of r. Call this function $d(r)$.

   Hint: $\frac{2}{1-t^2} = \frac{1}{1-t} + \frac{1}{1+t}$.

2. Solve for r in terms of d. Prove that $r = \frac{e^d - 1}{e^d + 1}$.

3. Suppose this curve is traversed at unit speed starting at the origin. Then in time t we must have traveled distance t. Find a formula for $\gamma(t)$ in terms of t. This γ is a curve with constant speed, and in class we showed that it is a geodesic. Show that at each finite time the curve is still inside the disk, and that it takes infinitely long to reach the boundary.

4. This disk can be parameterized using polar coordinates, and in that case

$$ds^2 = 4\frac{dr^2 + r^2 d\theta^2}{(1 - r^2)^2}.$$

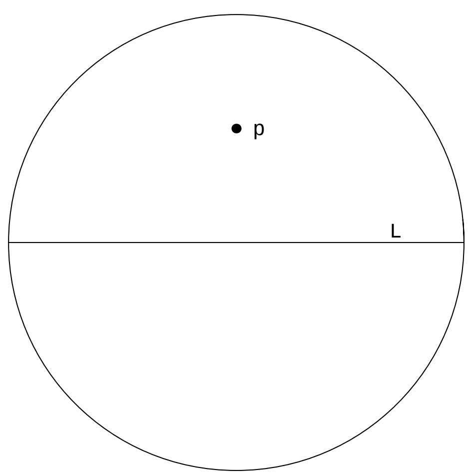
Write down the g_{ij}, which can just be read off from this formula. Then compute the Γ_{ij}^k.

5. Write down the differential equations satisfied by geodesics in non-Euclidean geometry, using this polar coordinate system.

6. The answer to the previous problem is two equations, one for $\frac{d^2r}{dt^2}$ and one for $\frac{d^2\theta}{dt^2}$. Show that the second equation is certainly solved if θ is constant. The resulting curve will be a line through the origin.

7. (This is the interesting exercise.) Suppose θ is constant. Show that the first equation just says that $||\gamma'(t)||$ is constant and thus requires that the radial line be traced at constant speed using the non-Euclidean notion of speed. Conclude that the curve given in exercise three solves the geodesic equation. If you wish, you can show this by substituting in and checking, but it also follows from the general principle of this problem.

8. Non-Euclidean geometry was first discovered synthetically, using the methods of Euclid. The starting point was the parallel postulate, which we can state in the following form: "given a geodesic l and a point p not on l, there is exactly one geodesic m through p which does not intersect l." In the non-Euclidean disk discussed in this exercise set, the geodesics are straight lines through the origin and circles which meet the boundary at ninety degrees. In the following picture, show that the parallel postulate is false by finding infinitely many geodesics through p parallel to L.



9. Drop a perpendicular from the point p to the line L. One of the first theorems proved synthetically stated that there is an angle α at p such that lines through p with angle less than α meet L, while lines with larger angle are parallel. See the picture below. There is a formula for α in terms of the non-Euclidean length d of the perpendicular from p to L. In particular, it turns out that $\alpha \to \frac{\pi}{2}$ as $d \to 0$ and $\alpha \to 0$ and $d \to \infty$. So when d is small, the world seems to be Euclidean, while as d gets large, the non-Euclidean nature of the plane becomes apparent. Find this formula for α in terms of d. Hint: this is partly a high school geometry problem. Find the centers of the circles which form the two limiting parallels through p.