Assignment 6. Due Monday, May 9.

Recall that the last two exercises from the previous assignment are due Monday. In addition, please do the following problems.

1. Consider the surface $z = x^3y^2$ with standard parameterization $s(x, y) = (x, y, x^3y^2)$. Let $\tilde{g}(x, y, z) = xy^2z$. Find the corresponding function $g(x, y)$ in the coordinate plane.

   Let X be the coordinate vector $(2, 3)$ at the point $(1, 1)$. Find the corresponding $\tilde{X}$ tangent to the surface at $\tilde{p} = (1, 1, 1)$. Compute $\tilde{X}(\tilde{g})$ at $\tilde{p}$. Compute $X(g)$ at p. The two results should be equal. Both answers should equal 20.

2. Switching to our new notation for vectors, let $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and let $Y = 2\frac{\partial}{\partial x}$. Compute the vector field $[X, Y]$.

3. Consider the cone parameterized in polar coordinates by $s(r, \theta) = (r\cos\theta, r\sin\theta, r)$. Let U be the vector field $U = (x, y, 0)$ on this surface. Notice that U is not tangent to the surface. The corresponding U in local coordinates is $U = (r\cos\theta, r\sin\theta, 0)$. This expression is still a vector in three dimensions, and its coordinates are the rectangular coordinates describing vectors in R^3. But their dependence on a point in the surface is written in polar coordinates.

   Compute the derivative of this vector with respect to $X = \frac{\partial}{\partial r}$ and $Y = \frac{\partial}{\partial \theta}$. Yes, this calculation is as easy as it looks.

   And now comes the hard part. Draw pictures of the vector field on the cone, and then convince me using English sentences that your results are reasonable.

4. Consider the saddle $z = xy$, parameterized as usual by $s(u, v) = (u, v, uv)$. Compute the unit normal vector field n to the saddle; notice that this field is not tangent to the saddle. Write the field in local coordinates $n(u, v)$. Compute the derivative of this field with respect to $X = \frac{\partial}{\partial u}$ and $Y = \frac{\partial}{\partial v}$ at the origin. Show that the resulting three dimensional vectors are tangent to the surface at this point. Show that $X(n)$ does not point in the direction of X and $Y(n)$ does not point in the direction of Y.

   Repeat the calculation for $X = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$ and $Y = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$. Show that the resulting vectors are tangent to the saddle, but even better, show that $X(n)$ is a multiple of X and $Y(n)$ is a multiple of Y. These multiples are the Euler curvatures κ_1 and κ_2.

5. Consider the surface $z = x^2 + y^2$ parameterized by $s(r, \theta) = (r\cos\theta, r\sin\theta, r^2)$. Compute the unit normal vector n; recall that it is $\frac{\partial s}{\partial r} \times \frac{\partial s}{\partial \theta}$ divided by its length. The answer has the form $n(r, \theta)$ but is a three dimensional vector. Show that the resulting normal is just $\frac{(-2x, -2y, 1)}{\sqrt{1 + 4(x^2 + y^2)}}$ if we use rectangular local coordinates.

Compute the derivatives of this vector with respect to r and with respect to θ. Said another way, compute the derivatives of n with respect to $X = \frac{\partial}{\partial r}$ and $Y = \frac{\partial}{\partial \theta}$. Show that $X(n)$ is a multiple of X and $Y(n)$ is a multiple of Y. These multiples are κ_1 and κ_2 up to sign.

Remark: The remaining exercises are for the graduate students, although undergraduates who are curious can certainly try them.

6. Let V be a finite dimensional vector space over R. The material which follows works for any field of characteristic not two. A bilinear form on V is an assignment to each v and w in V of a number $b(v, w)$, which is linear in v if w is held fixed and linear in w if v is held fixed. A linear transformation from V to V is a map $B : V \to V$ which satisfies — well, you know.

   Suppose $e_1, \ldots, e_n$ is a basis for V. Then both b and B give rise to matrices. Indeed $b_{ij} = b(e_i, e_j)$ and $B(e_i) = \sum_j B_{ji} e_j$. Let $X = X_1 e_1 + \ldots + X_n e_n$ and $Y = Y_1 e_1 + \ldots + Y_n e_n$. Then
   $$b(X, Y) = \sum b_{ij} X_i Y_j$$
   $$B(X)_i = \sum B_{ij} X_j$$
   It is tempting to conclude that bilinear forms, linear transformations, and matrices are all the same thing. But there is a problem. Suppose $f_1, \ldots, f_n$ is a new basis and $f_i = \sum a_{ji} e_j$. Show that the new matrices for b and B are $A^T b A$ and $A^{-1} B A$. Thus the matrix associated with a bilinear form transforms differently than the matrix associated with a linear transformation.

7. Sometimes people first learn linear algebra in coordinates. This can be dangerous, because ideas which apply to matrices don't automatically apply to bilinear forms or to linear transformations. Show that $\det(B)$ depends only on the linear transformation and is independent of the choice of basis. Show by example that $\det(b)$ is not independent of the basis and thus does not really make sense for bilinear forms.

8. The eigenvalues of a matrix are computed as zeros of $P(\lambda) = \det(\lambda I - B)$. These eigenvalues make sense for linear transformations since they also have an invariant definition: $Bv = \lambda v$ for some $v \neq 0$. But show that eigenvalues do not make sense for b. Construct a b and a λ which is an eigenvalue of the matrix associated to b in one coordinate system, but not an eigenvalue of the matrix associated to b in another coordinate system.

9. However, if you also have an inner product $< \star, \star >$ on V, then we can map bilinear forms to linear transformations. Let b be a bilinear form on V. Then b defines a linear transformation $\varphi : V \to V^\star$ from V to its dual space by $v \to \varphi_v$ where $\varphi_v(w) = b(v, w)$. In particular, $< \star, \star >$ is a bilinear form, so it defines a linear transformation $\psi : V \to V^\star$. Show that this map is an isomorphism.

10. Given a bilinear form b and an inner product $< \star, \star >$, we can construct a linear transformation $B : V \to V$ by composing the two maps of the previous exercise: $\psi^{-1} \circ \varphi : V \to V^\star \to V$. Show that this B is defined by the equation

$$b(X, Y) =< B(X), Y >$$

Show that if we choose an *orthonormal* basis, then $b_{ij} = B_{ji}$.

11. We say a bilinear form is *symmetric* if $b(X, Y) = B(Y, X)$. Show that this is equivalent to the equation

$$< B(X), Y >=< X, B(Y) >$$

In turn, show that this is equivalent to the matrix condition $B^T = B$ for the matrix, provided that the matrix is obtained from an *orthonormal* coordinate system.

12. A fundamental result in linear algebra states that if B is symmetric, then there is an orthonormal basis $e_1, \ldots, e_n$ of eigenvectors of B. Write $B(e_i) = \kappa_i e_i$. Show that in this coordinate system, $b(X, X) = \sum \kappa_i X_i^2$.

Remark: The above theory applies immediately to differential geometry. We have a natural b*, the second fundamental form. We also have a natural* $< \star, \star >$*, the metric tensor or first fundamental form. The exercises show that it is dangerous to talk about "the eigenvalues of* b*", but it is legal to discuss then in the presence of* $< \star, \star >$ *. These eigenvalues are exactly the Euler curvatures of the surface. As we'll see, the formulas for* b *do not involve the* g_{ij}*, but when we compute Euler curvature and later Gaussian curvature, the* g_{ij} *reappear. These exercises explain why.*