Assignment 7. Due Monday, May 16.

1. The $\mathcal{S}$ be a surface, $p \in \mathcal{S}$. For each nonzero tangent vector $X \in T_p$, consider the expression $K(X) = \frac{b(X,X)}{<X,X>}$. Prove that this only depends on the direction of X and not on its length. That is, prove that the expression does not change if we replace X by λX where $\lambda \neq 0$.

   Later in the course, we will prove that if $\gamma(t)$ is a geodesic with $\gamma(0) = p$ and $\gamma'(0) = X$, then the above expression is the curvature of γ.

2. Choose an orthonormal basis e_1, e_2 for T_p consisting of eigenvectors of B. Indeed, suppose $B(e_1) = -\kappa_1 e_1$ and $B(e_2) = -\kappa_2 e_2$. By the previous exercise, $K(X)$ only depends on the angle θ between X and e_1. Show that

   $$K(X) = \frac{\kappa_1 - \kappa_2}{2} \cos 2\theta + \frac{\kappa_1 + \kappa_2}{2}$$

   Assume that $\kappa_2 \leq \kappa_1$ and draw a graph of K as a function of θ. Notice that the maximum of this function is κ_1 when X points in the direction of e_1 and the minimum is κ_2 when X points in the direction of κ_2. Many authors define $\kappa_1(p)$ and $\kappa_2(p)$ as the maximum and minimum curvatures of geodesics at p.

3. Consider the surface $z = x^2 + y^2$ parameterized by $s(r, \theta) = (r \cos\theta, r \sin\theta, r^2)$. This surface was studied in assignment five. You may wish to look at the solutions available on the web. These solutions show that at any point (r, θ), the transformation $B(v) = v(n)$ has eigenvectors $X = \frac{\partial}{\partial r}$ and $Y = \frac{\partial}{\partial \theta}$ with corresponding $\kappa_1 = \frac{1}{(1+4r^2)^{3/2}}$ and $\kappa_2 = \frac{2}{\sqrt{1+4r^2}}$. Explain why X is tangent to an obvious geodesic. Write down this geodesic as a parameterized curve (not necessarily parametrized by arc length) and show that its curvature is κ_1.

   The vector Y is tangent to the circle of constant r around the paraboloid. This circle has curvature $\frac{1}{r}$. Explain why the number $\kappa_2 = \frac{2}{\sqrt{1+4r^2}}$ is approximately $\frac{1}{r}$ but actually slightly less.

   The circle just examined is not a geodesic. Draw a picture of the geodesic tangent to Y and explain in English why its curvature should be slightly less than $\frac{1}{r}$.

4. Consider the torus parameterized by

   $$s(\phi, \theta) = (R \cos\theta, R \sin\theta) + r \cos\phi(\cos\theta, \sin\theta, 0) + r \sin\phi(0, 0, 1)$$

   Find the principal curvatures κ_1 and κ_2 as functions of ϕ and θ. Actually by symmetry these functions should depend only on ϕ. Explain why.

Hint: You can do this problem two ways. You can directly calculate, which will be messy. Or you can consult pages 82 - 84 of the notes, where the principal curvatures of a surface of revolution are calculated. Notice that a torus can be obtained by rotating a circle of radius r centered at $(0, R)$ about the x-axis. You may want to consider the upper half and the lower half of this circle separately.

This torus has two kinds of obvious geodesics. One circles the solid part of the torus with radius r. Show that the curvature of this geodesic is one of the κ_i. The other circles the inside and outside of the torus with radii $R - r$ and $R + r$. Show that these circles give the other κ_i at certain special points.

The Remaining Problems Are For The Graduate Students

5. There is no need to restrict attention to surfaces embedded in R^3. We can consider $s(u, v) : \mathcal{U} \subseteq R^2 \to R^n$ and compute the g_{ij} exactly as before. Our extrinsic theory now fails, so we cannot compute κ_1 and κ_2. But by later developments in the course, which you should assume, we can compute κ corresponding to the product $\kappa_1\kappa_2$ for surfaces embedded in R^3.

   The parametrization of the torus in an earlier problem mapped the square

   $$\{ \ (\phi, \theta) \mid 0 \le \phi \le 2\pi \text{ and } 0 \le \theta \le 2\pi \ \}$$

   to a torus in R^3. Thus it set up the well-known correspondence between a torus and a square with opposite sides identified.

   There is an even easier map from the square with opposite points identified to a torus in R^4. Namely, think of R^4 as $C \times C$ and map (ϕ, θ) to $\left(e^{i\phi}, e^{i\theta}\right)$. Show that if we use this mapping, we get $g_{ij} = \delta_{ij}$. Hence the geometry on the resulting torus is ordinary Euclidean geometry.

   In applications, this torus often comes up naturally. It is called the *flat torus*.

6. Explain why no embedding of the torus into R^3 will give the flat metric. I don't need a precise proof if you can give a convincing idea. Hint: for the flat torus, $\kappa = 0$. Show that $\kappa = \kappa_1\kappa_2$ cannot be identically zero on a compact surface embedded in R^3.

7. It is natural to wonder if other compact surfaces can be embedded in a high dimensional R^n such that the induced geometry is Euclidean. Show that only the torus has this property. Here too, a sketch of the proof will suffice, and I recommend using the hint below.

   Hint: If the induced metric is Euclidean, then each point has local coordinates in which triangles with geodesic sides have angle sum π. Triangulate the entire surface using these triangles. Then compute the sum over all triangles of the angle sum of the triangle. Show that it is π times the number of triangles. Show that it is also 2π

times the number of vertices. Show that the number of edges is $\frac{3}{2}$ times the number of triangles. Compute the Euler characteristic of the surface.