

## Assignment 8. Due Monday, May 23.

1. Let  $U$  be a three dimensional vector field defined a surface but not necessarily tangent to the surface. Let  $\gamma(t) = (u(t), v(t))$  be a path in the coordinate plane; this path induces a corresponding path  $\gamma(t)$  on the surface.

Since  $\gamma'(t)$  is a tangent vector, we can compute the derivative of  $U$  in the direction  $\gamma'(t)$ . The notation for this derivative is  $\gamma'(t)(U)$ .

There is a very easy way to compute this derivative. At each time  $t$ ,  $\gamma(t)$  is a point on the surface and  $U$  at this point is an ordinary three dimensional vector. Write this vector  $U(t) = (U_1(t), U_2(t), U_3(t))$ .

Prove that

$$\gamma'(t)(U) = \frac{dU}{dt} = \left( \frac{dU_1}{dt}, \frac{dU_2}{dt}, \frac{dU_3}{dt} \right)$$

2. Now suppose that each  $U(t)$  is tangent to the surface. To emphasize this point, we will call this vector field  $Y$ . In class we decomposed  $X(Y)$  into a normal and a tangential component:  $X(Y) = b(X, Y)n + \nabla_X Y$ . Show that when we do this with  $\frac{dY}{dt}$  the decomposition is

$$\frac{dY}{dt} = b(\gamma', Y)n + \frac{DY}{dt}$$

where by definition  $\frac{DY}{dt}$  is the tangent vector whose  $k$ th component is given by

$$\left. \frac{DY}{dt} \right|_k = \frac{dY_k}{dt} + \sum_{ij} \Gamma_{ij}^k \frac{d\gamma_i}{dt} Y_j(t)$$

3. In particular, show that the geodesic differential equation is just the statement that the tangential component of acceleration is zero, that is,  $\frac{D\gamma'}{dt} = 0$ . Equivalently, it is the statement that  $\frac{d^2\gamma}{dt^2}$  is normal to the surface.
4. Suppose  $\gamma(t)$  is a geodesic. Then  $\gamma(t)$  has constant speed; suppose that actually this constant speed is one, so  $\gamma$  is parameterized by arc length. Show that  $n$  is the normal  $N$  to  $\gamma$  which we discussed in the first part of this course, and show that the curvature of  $\gamma$  is  $\kappa = b(\gamma', \gamma')$ .
5. If  $X$  is any nonzero tangent vector at a point  $p$ , not necessarily of length one, show that  $\frac{b(X, \ddot{X})}{\langle X, X \rangle}$  is the curvature of the unique geodesic  $\gamma(t)$  with  $\gamma(0) = p$  and  $\gamma'(0) = X$ .

*Remark for Graduate Students:* There is a subtle point which I want you to think about. In the exercises, we differentiate  $\gamma'(t)$  in the direction  $t$ . But this  $\gamma'$  is not defined everywhere on the surface. It is only defined on the curve. So the exercises don't quite make sense, starting with exercise three.

Indeed,  $\gamma(t)$  might cross itself, so  $\gamma(t_1) = \gamma(t_2)$ , and the curve might not be tangent to itself when it crosses, so  $\gamma'(t_1) \neq \gamma'(t_2)$ . Thus we couldn't make the exercises rigorous by just extending  $\gamma'$  to a vector field on the entire surface.

We can fix this problem in several ways. Suppose  $U(t)$  is an assignment to each time  $t$  of a three dimensional vector starting at  $\gamma(t)$ . If  $\gamma(t_1) = \gamma(t_2)$ , there is no requirement that  $U(t_1) = U(t_2)$ . In this case we *define*  $\frac{dU}{dt}$  by the first displayed formula on the previous page.

The main goal is then to prove the decomposition law in exercise 2. At points where  $\gamma' = 0$  this law is easy to prove. At other points, we can locally extend  $\gamma'$  to a vector field by the implicit function theorem, and then give the proof in the exercises. So the result holds in general.

If  $U$  is tangent to the surface at each time, we call it  $Y(t)$  and then *define*  $\frac{DY}{dt}$  by the equation on the previous page. This expression is independent of coordinates because it has an invariant interpretation given by exercise two.