Assignment 9. Due Wednesday, June 1.

In this homework I'm going to ask you to prove one of the most beautiful results from the geometry of surfaces. The result is also proved in the lecture notes, but my proof is way too complicated. (So I'm asking you to rewrite the notes.) In particular, the lecture notes use a complicated formula for the curvature tensor R_{ijkl} in terms of derivatives of the Christoffel symbols, which you can avoid.

The proof is a nice illustration of the abstract ideas we have been developing since the midterm, and not too difficult. But first...

Let me state the result and put it into context. According to Riemann's counting argument, the Gaussian curvature κ should completely describe the geometry of a curved surface. But this is a sort of "moral principle" rather than a precise theorem. Indeed, κ is a function rather than a number, and it is difficult to compare κ's on different surfaces.

In practice, Riemann's argument tells us that generalizations of results in classical Euclidean geometry to surfaces will probably require understanding the role of κ in the generalization. The last theorem of our course, due to Gauss and Bonnet, is an example. We will generalize the Euclidean theorem that the sum of the angles of a triangle is 180 degrees; the generalization has an extra term involving the integral of κ over the triangle.

There is one special case, however, where we can convert Riemann's argument into a precise theorem. It is the case when κ is a constant. We then have the following results:

Theorem 1 *Suppose κ is constant on a surface. Magnify the surface by M. On the new surface, $< X, Y >$ is replaced by $M^2 < X, Y >$ and g_{ij} is replaced by $M^2 g_{ij}$. Then the new surface has constant curvature $\frac{\kappa}{M^2}$.*

It follows from this theorem that it suffices to understand surfaces with constant Gaussian curvature κ equal to $1, 0$, or -1.

Theorem 2

1. *the Gaussian curvature of the sphere of radius one is $\kappa = 1$;*
2. *the Gaussian curvature of the plane is $\kappa = 0$;*
3. *the Gaussian curvature of the Poincare model of non-Euclidean geometry is $\kappa = -1$.*

Definition 1 *Let $p \in \mathcal{S}_1$ and $q \in \mathcal{S}_2$ be points in two surfaces. We say the surfaces are locally isometric near p and q is there are open neighbors $\mathcal{U}$ and $\mathcal{V}$ of p and q and a one-to-one, onto, $\mathcal{C}^\infty$ map from $\mathcal{U}$ to $\mathcal{V}$ with $\mathcal{C}^\infty$ inverse, such that the metric tensor of $\mathcal{U}$ is mapped to the metric tensor of $\mathcal{V}$. Intuitively, then, the surfaces have the same geometry near p and q.*

Theorem 3 *If $\mathcal{S}_1$ an $\mathcal{S}_2$ are locally isometric near p and q, then $\kappa(p) = \kappa(q)$. Conversely, if κ is constant on both surfaces and the constants are equal, then the surfaces are locally isometric near p and q.*

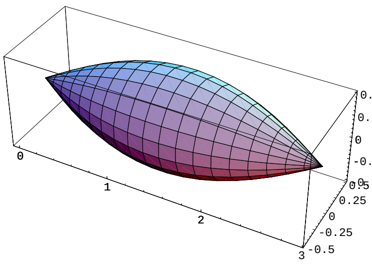
Definition 2 *We say a surface is locally homogeneous if for any $p \in \mathcal{S}$ and $q \in \mathcal{S}$, the surface is locally isometric near p and q. Intuitively, a surface is locally homogeneous if the geometry near any point is the same as the geometry near any other point.*

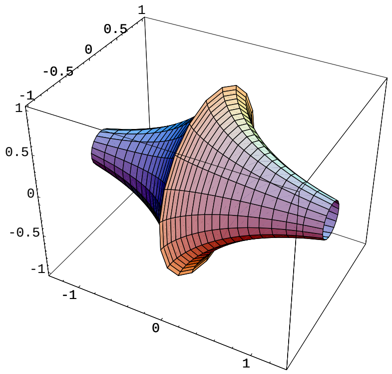
Corollary 4 *A surface is locally homogeneous if and only if it has constant Gaussian curvature.*

Corollary 5 *Any surface with constant Gaussian curvature can be magnified to become locally isometric to either the sphere of radius one, the plane, or the Poincare model of non-Euclidean geometry.*

Remark: Thus the main result of this exercise set is: **There are exactly three two-dimensional geometries up to magnification, if we require that a geometry look the same from any point p.**

Remark: This theorem does *not* say that the only surfaces with constant curvature are the three standard models. There are lots of surfaces in R^3 with constant curvature. For example, take any piece of paper and fold it without crinkling into a cone, cylinder, or twisted sheet of some sort. All of these surfaces have curvature zero and are locally isometric to the plane. Below are two surfaces of constant curvature one and minus one. These surfaces don't look locally homogenious; the portion near the ends looks different than the fat portion in the middle. Indeed the principal curvatures κ_1 and κ_2 are different in the fat portion and the end portion. But κ is the same in both portions, and the geometry as seen by a two dimensional person is the same in both portions.





There are many other surfaces of constant curvature. Two dimensional people living on these surfaces think the geometry is exactly like that of the sphere (or the plane, or the Poincare model) as long as they stay close to home. When they wander like topologists, then they see the difference.

As a special case of these results, we have

Theorem 6 *We can choose coordinates near each point p such that $g_{ij} = \delta_{ij}$ if an only if the Riemann curvature tensor R_{ijkl} is identically zero.*

Remark: It is easy to see that all R_{ijkl} are zero in one coordinate system if and only if they are all zero in any other coordinate system. Physicists say that this the characteristic property of a *tensor.* Thus the Riemannian R_{ijkl} is a tensor. So to decide if our geometry is Euclidean, we can pick *any* coordinate system and compute R_{ijkl}. Our geometry is Euclidean exactly if this tensor is zero.

The Christoffel symbols Γ_{ij}^k do not have this property. If we choose standard Euclidean coordinates for the plane, then the $\Gamma_{ij}^k = 0$. But in polar coordinates we computed the Γ_{ij}^k and they were not all zero. Physicists say that Γ_{ij}^k is not a tensor. The Christoffel symbols are useless if we want to determine whether the geometry is Euclidean because even if they are nonzero the geometry may be Euclidean. But the R_{ijkl} are different; we need only compute them in *some* coordinate system to determine if our geometry is Euclidean.

Finally, here are the exercises. They are easier than the introduction!

1. Let $s(u, v)$ parameterize a surface $\mathcal{S} \subset R^3$. Fix a magnification constant M and define a new surface by $Ms(u, v)$. This is just the original surface magnified by M. Show that the new $\tilde{g}_{ij}$ equal the original g_{ij} multiplied by M^2: $\tilde{g}_{ij} = M^2 g_{ij}$. Another way to say this is that the new $< X, Y >$ is the original $< X, Y >$ multiplied by M^2. Write down our formula for Γ_{ij}^k and explain why the new Γ_{ij}^k and the original Γ_{ij}^k are equal. Conclude that $\nabla_X Y$ is the same for the new surface and the original surface.

   Using this fact, show that the new $R(X, Y, Z, W)$ is M^2 times the original one: $\tilde{R}(X, Y, Z, W) = M^2 R(X, Y, Z, W)$. To find κ, we are supposed to find an orthonormal basis f_1, f_2 and compute $R(f_1, f_2, f_1, f_2)$. Notice that an orthonormal basis for the old s is no longer orthonormal for the new s. If f_1, f_2 is orthonormal for the old s, show that $\frac{f_1}{M}$ and $\frac{f_2}{M}$ is orthonormal for the new s. Conclude that the Gaussian curvature for the new s is $\frac{\kappa}{M^2}$ where κ is the curvature for the old s.

2. Consider $\mathcal{S} = R^2$ with its standard Euclidean metric $g_{ij} = \delta_{ij}$. Briefly explain why $\Gamma_{ij}^k = 0$. Using this fact, write down a formula for $\nabla_{\frac{\partial}{\partial u_i}} (Y_1, Y_2)$. Then write down a formula for $\nabla_{\frac{\partial}{\partial u_i}} \nabla_{\frac{\partial}{\partial u_j}} (Y_1, Y_2)$. Explain why R is identically zero, and thus why $\kappa = 0$. All this should be easy.

3. We now compute κ on a sphere of radius one. You could do this using the formula $\kappa = \kappa_1 \kappa_2$ since we proved that $\kappa_1 = \kappa_2 = 1$ on the "cheat sheet" passed out in class after the last midterm. Instead, use the hints below together with formulas on the

cheat sheet to prove $\kappa = 1$ directly.

Let $e_1 = \frac{\partial}{\partial \theta}$ and $e_2 = \frac{\partial}{\partial \phi}$ as on the sheet. Recall that

$$R(e_1, e_2, e_1, e_2) = \det(g_{ij}) R(f_1, f_2, f_1, f_2) = -\det(g_{ij}) \kappa_1 \kappa_2$$

where f_1, f_2 is an orthonormal basis. Using the cheat sheet, conclude that

$$R(e_1, e_2, e_1, e_2) = -(\sin^2 \phi)\kappa$$

To show that $\kappa = 1$ it suffices to compute

$$< \nabla_{e_1} \nabla_{e_2} e^1 - \nabla_{e_2} \nabla_{e_1} e^1 - \nabla_{[e_1, e_2]} e_1, e_2 >$$

Explain why $[e_1, e_2] = 0$. Then use the cheat sheet to compute the displayed expression above. Show that the answer is $-\sin^2 \phi$ and so $\kappa = 1$.

Conclude that κ on a sphere of radius R is $\frac{1}{R^2}$.

4. Repeat this calculation for Poincare's model of non-Euclidean geometry. Recall that

$$g_{ij} = \frac{4(dx^2 + dy^2)}{(1 - (x^2 + y^2))^2}$$

Convert to polar coordinates, so

$$g_{ij} = \frac{4(dr^2 + r^2 d\theta^2)}{(1 - r^2)^2}$$

Of course this formula means

$$g_{11} = \frac{4}{(1 - r^2)^2} \qquad g_{12} = 0 \qquad g_{22} = \frac{4r^2}{(1 - r^2)^2}$$

Notice that we cannot compute Gaussian curvature as $\kappa_1 \kappa_2$ because there is no surface $\mathcal{S} \subset R^3$ giving this geometry. We must compute using the Riemann curvature tensor. The number 4 will become important for this first time; if it were absent, we wouldn't get $\kappa = -1$.

Compute κ in exactly the same way it was computed for the sphere. First let $e_1 = \frac{\partial}{\partial r}$ and $e_2 = \frac{\partial}{\partial \theta}$. Show that

$$R(e_1, e_2, e_1, e_2) = -\frac{16r^2}{(1 - r^2)^4} \kappa$$

Then explain why

$$< \nabla_{e_1} \nabla_{e_2} e_1 - \nabla_{e_2} \nabla_{e1} e_1, e_2 >= -\frac{16r^2}{(1 - r^2)^4} \kappa$$

Show that this formula is the same as

$$< \nabla_{e_1} \left(\Gamma^1_{12} e_1 + \Gamma^2_{12} e_2 \right) - \nabla_{e_2} \left(\Gamma^1_{11} e_1 + \Gamma^2_{11} e_2 \right), e_2 >= -\frac{16r^2}{(1-r^2)^4} \kappa$$

Explain why the only nonzero Christoffel symbols are $\Gamma^1_{11}, \Gamma^1_{22}$, and Γ^2_{12}. Explain why Christoffel symbols depend only on r and not on θ, so their derivatives with respect to e_2 are zero. Explain why the previous expression reduces to

$$\frac{\partial}{\partial r} \left(\Gamma^2_{12} \right) < e_2, e_2 > + \Gamma^2_{12} < \nabla_{e_1} e_2, e_2 > - \Gamma^1_{11} < \nabla_{e_2} e_1, e_2 >= -\frac{16r^2}{(1-r^2)^4} \kappa$$

and explain why this further reduces to

$$\frac{\partial}{\partial r} \left(\Gamma^2_{12} \right) < e_2, e_2 > + \left(\Gamma^2_{12} \right)^2 < e_2, e_2 > - \Gamma^1_{11} \Gamma^2_{12} < e_2, e_2 >= -\frac{16r^2}{(1-r^2)^4} \kappa$$

and therefore to

$$\frac{\partial}{\partial r} \left(\Gamma^2_{12} \right) + \left(\Gamma^2_{12} \right)^2 - \Gamma^1_{11} \Gamma^2_{12} = -\frac{4}{(1-r^2)^2} \kappa$$

Finally, compute Γ^2_{12} and Γ^1_{11} and the left side of this equation, and show that $\kappa = -1$.

5. **The remaining exercise is for the graduate students. Undergraduates may try it if they wish!** Suppose κ is a constant equal to $1, 0$, or -1 and $\mathcal{S}$ is a surface with constant Gaussian curvature κ. Eventually you will show that in a neighborhood of any $p \in \mathcal{S}$ there are coordinates u, v such that $g_{11} = h^2(u), g_{12} = 0, g_{22} = 1$, where $h(u) = \cos u$ if $\kappa = 1$, $h(u) = 1$ if $\kappa = 0$, and $h(u) = \cosh u$ if $\kappa = -1$. Explain in a sentence why all of the remaining claims at the start of this exercise set follow from this result.

   It remains to construct these coordinates. We'll give a general argument. To visualize the argument, imagine that our surface is a sphere and that p is a point on the equator.

   Draw a geodesic starting at p with constant speed 1. Suppose this geodesic is defined for $|u| < \epsilon$. Think of distances along this geodesic as giving the $u-$coordinate. On the sphere you can think of this geodesic as the equator.

   At each point of the u-geodesic, draw a geodesic starting in a perpendicular direction, also of constant speed 1. Think of distances along these geodesics as giving the v-coordinate. On the sphere, these geodesics are lines of constant longitude. Imagine that each such geodesic is defined for $|v| < \epsilon$.

   Assign coordinates (u, v) to points q in the $\epsilon \times \epsilon$ box around p by moving along the original geodesic by u until you come to the perpendicular geodesic through q, and then moving along this perpendicular geodesic by v to get to q.

Notice that the coordinate curve with constant $v = v_0$ can be drawn by moving along each perpendicular geodesic a distance v_0 and then connecting all of these points together to form a curve. This curve need not be a geodesic, just as on the sphere the lines of constant latitude are not geodesics.

Let $e_1 = \frac{\partial}{\partial u}$ and $e_2 = \frac{\partial}{\partial v}$. Incidentally, if you are reading the notes where I prove this theorem, you should notice that I have interchanged the roles of u and v.

Notice first that the lines of constant u in this coordinate system are geodesics of speed one. Explain why it follows that $g_{22} = 1$ and $\nabla_{e_2} e_2 = 0$.

Next we want to prove that $g_{12} = 0$. Thus we want to prove that $< e_1, e_2 >$ is identically zero. Explain why this expression equals zero on the original geodesic when $v = 0$. To prove it identically zero, it suffices to prove that its derivative in the v direction is identically zero, and thus that $e_2 < e_1, e_2 >= 0$. Prove that this expression is $< \nabla_{e_2} e_1, e_2 > + < e_1, \nabla_{e_2} e_2 >$. Then prove that it is $< \nabla_{e_2} e_1, e_2 >$. Prove that this equals $< \nabla_{e_1} e_2, e_2 >$, which in turn equals $\frac{1}{2} e_1 < e_2, e_2 >= 0$.

Notice that g_{11} is consequently positive and so it can be written $h^2(u, v)$ for some h. Thus $g_{11} = h^2(u, v), g_{12} = 0, g_{22} = 1$. It follows that we can introduce coordinates of this type of *any surface* since we haven't yet used the hypothesis that κ is constant.

To finish, we need to use the hypothesis that κ is constant. Explain why

$$< \nabla_{e_1} \nabla_{e_2} e_1 - \nabla_{e_2} \nabla_{e_1} e_1, e_2 >= -h^2 \kappa.$$

Explain why the only nonzero Christoffel symbols are $\Gamma^1_{11}, \Gamma^2_{11}$, and Γ^1_{12}. Explain why the displayed equation is equivalent to the equation

$$\Gamma^1_{12} \Gamma^2_{11} - \frac{\partial}{\partial v} \left(\Gamma^2_{11} \right) = -h^2 \kappa$$

Compute the Christoffel symbols and show that the previous equation is equivalent to the equation

$$- \left(\frac{\partial h}{\partial v} \right)^2 - \frac{\partial}{\partial v} \left(-h \frac{\partial h}{\partial v} \right) = -h^2 \kappa$$

and so

$$\frac{\partial^2 h}{\partial v^2} = -h \kappa$$

Hence $h(u, v)$ is given by

$$\begin{array}{rl} A(u) \cos v + B(u) \sin v & \text{if } \kappa = 1 \\ A(u) v + B(u) & \text{if } \kappa = 0 \\ A(u) \cosh v + B(u) \sinh v & \text{if } \kappa = -1 \end{array}$$

Finally, explain why e_1 has length one on the original geodesic when $v = 0$. Conclude that $h(u, 0) = 1$. Moreover, explain why $2h\frac{\partial h}{\partial v}$ must equal $e_2 < e_1, e_1 >$. Explain why this equals $2 < \nabla_{e_2} e_1, e_1 >= 2 < \nabla_{e_1} e_2, e_1 >$. Compare this with $e_1 < e_2, e_1 >= < \nabla_{e_1} e_2, e_1 > + < e_2, \nabla_{e_1} e_1 >$ and conclude that the two expressions are equal up to a factor of 2 when $v = 0$. Show that the second expression is identically zero, and conclude that $\frac{\partial h}{\partial u}(u, 0) = 0$. Conclude that

$$\begin{array}{rl} \cos v & \text{if } \kappa = 1 \\ 1 & \text{if } \kappa = 0 \\ \cosh v & \text{if } \kappa = -1 \end{array}$$