

The Entire Theory in the Case of the Sphere

Consider the sphere with spherical coordinates $(\theta, \phi) \rightarrow (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$. Spherical coordinates differ from standard longitude and latitude conventions in one small way: increasing ϕ moves down toward the equator rather than up toward the north pole.

Standard Formulas

$$e_1 = (1, 0) = \frac{\partial}{\partial \theta} \leftrightarrow \frac{\partial s}{\partial \theta} = (-\sin \phi \sin \theta, \sin \phi \cos \theta, 0)$$

$$e_2 = (0, 1) = \frac{\partial}{\partial \phi} \leftrightarrow \frac{\partial s}{\partial \phi} = (\cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi)$$

$$n = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$$

$$g_{11} = \sin^2 \phi$$

$$g_{12} = 0$$

$$g_{22} = 1$$

Derivatives of Basis Vector Fields

$$e_1(e_1) = \frac{\partial}{\partial \theta} (-\sin \phi \sin \theta, \sin \phi \cos \theta, 0) = (-\sin \phi \cos \theta, -\sin \phi \sin \theta, 0)$$

$$e_1(e_2) = (-\cos \phi \sin \theta, \cos \phi \cos \theta, 0)$$

$$e_2(e_1) = (-\cos \phi \sin \theta, \cos \phi \cos \theta, 0)$$

$$e_2(e_2) = (-\sin \phi \cos \theta, -\sin \phi \sin \theta, -\cos \phi)$$

Decomposition: Now we decompose these derivatives into normal and tangential components. Notice that the basic equation below implies that $b(X, Y) = X(Y) \cdot n$.

$$X(Y) = b(X, Y)n + \nabla_X Y.$$

$$b(e_1, e_1) = (-\sin \phi \cos \theta, -\sin \phi \sin \theta, 0) \cdot (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) = -\sin^2 \phi$$

$$b(e_1, e_2) = 0$$

$$b(e_2, e_1) = 0$$

$$b(e_2, e_2) = -1$$

B Matrix: Recall that $b(X, Y) = - \langle B(X), Y \rangle$. The dot product of $B(X)$ with respect to the basis vectors completely determines $B(X)$, so these formulas can be used to read off $B(e_1)$ and $B(e_2)$.

$$\langle B(e_1), e_1 \rangle = \sin^2 \phi$$

$$\langle B(e_1), e_2 \rangle = 0$$

$$\langle B(e_2), e_1 \rangle = 0$$

$$\langle B(e_2), e_2 \rangle = 1$$

Principal Curvatures: It follows that $B(e_1) = e_1$ and $B(e_2) = e_2$. (Notice that the term $\sin^2 \phi$ comes from the fancy inner product.) So $\kappa_1 = \kappa_2 = -1$. This sign arises because the normal points outward but the principal curvatures are both inward toward the center of the sphere.

Tangential Components: We can read off the tangential components of derivatives using the formula

$$\nabla_X Y = X(Y) - b(X, Y)n$$

$$\nabla_{e_1} e_1 = (-\sin \phi \cos \theta, -\sin \phi \sin \theta, 0) + \sin^2 \phi (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$$

$$= (-\sin \phi \cos \theta \cos^2 \phi, -\sin \phi \sin \theta \cos^2 \phi, \sin^2 \phi \cos \phi) = -\sin \phi \cos \phi (\cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi) =$$

$$= -\sin \phi \cos \phi e_2$$

$$\nabla_{e_1} e_2 = \frac{\cos \phi}{\sin \phi} e_1$$

$$\nabla_{e_2} e_1 = \frac{\cos \phi}{\sin \phi} e_1$$

$$\nabla_{e_2} e_2 = 0$$

Christoffel Symbols: Only derivatives with respect to the second of (θ, ϕ) are nonzero, and only when differentiating $g_{11} = \sin^2 \phi$. So the only nonzero terms are

$$\Gamma_{11}^2 = \frac{1}{2} \frac{1}{g_{22}} \left(-\frac{\partial g_{11}}{\partial \phi} \right) = -\sin \phi \cos \phi$$

$$\Gamma_{12}^1 = \frac{1}{2} \frac{1}{g_{11}} \left(\frac{\partial g_{11}}{\partial \phi} \right) = \frac{\cos \phi}{\sin \phi}$$

Geodesic Equations: In general these equations are

$$\frac{d^2 \gamma_k}{dt^2} + \sum_{ij} \Gamma_{ij}^k \frac{d\gamma_i}{dt} \frac{d\gamma_j}{dt} = 0$$

In the case of the sphere these equations become

$$\frac{d^2 \theta}{dt^2} + \frac{2 \cos \phi}{\sin \phi} \frac{d\theta}{dt} \frac{d\phi}{dt} = 0$$

$$\frac{d^2 \phi}{dt^2} - \sin \phi \cos \phi \left(\frac{d\theta}{dt} \right)^2 = 0$$

Notice that when θ is constant, the equations are solved by $\phi(t) = at + b$, but if ϕ is constant, the equations are solved by $\theta(t) = at + b$ only when $\sin \phi \cos \phi = 0$, and thus only at the equator and the north and south poles.

Intrinsic Tangential Calculation: We can also compute $\nabla_X Y$ intrinsically using the Christoffel symbols. In this calculation,

$$\frac{\partial}{\partial u_i} (Y_1, Y_2) = \left(\frac{\partial Y_1}{\partial u_i}, \frac{\partial Y_2}{\partial u_i} \right) + (\Gamma_{i1}^1 Y_1 + \Gamma_{i2}^1 Y_2, \Gamma_{i1}^2 Y_1 + \Gamma_{i2}^2 Y_2)$$

When we use this formula to differentiate $e_1 = (1, 0)$ and $e_2 = (0, 1)$, the partial derivative portion vanishes and only the Christoffel portion remains:

$$\nabla_{e_1} e_1 = -\sin \phi \cos \phi e_2$$

$$\nabla_{e_1} e_2 = \frac{\cos \phi}{\sin \phi} e_1$$

$$\nabla_{e_2} e_1 = \frac{\cos \phi}{\sin \phi} e_1$$

$$\nabla_{e_2} e_2 = 0$$