Math 433 Outline for Final Examination

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1 Curves

From the chapter on curves, you should know

1. the formula for arc length of a curve;

2. the definition of $T(s), N(s), B(s)$, and $\kappa(s)$ for a curve parameterized by arc length;

3. how to find κ, but not τ, for a curve which is not parameterized by arc length;

4. the statement of the Frenet-Serret formulas, but not the proof

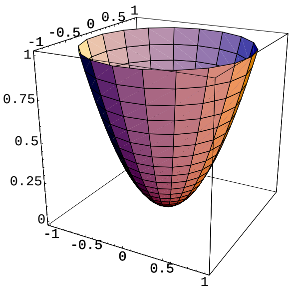
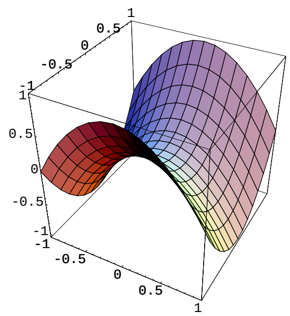
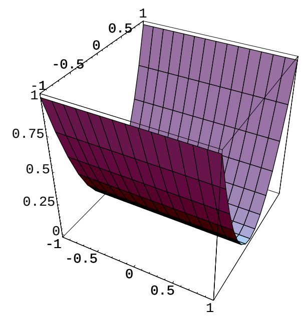
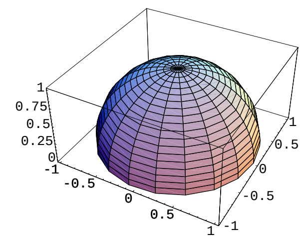
5. a method of constructing $\gamma(u)$ from $\kappa(u)$ and $\tau(u)$, for instance as given by the computer technique on page 14 of the notes

It is not necessary to understand how we reparameterize curves. Other sections of chapter one can be skipped.

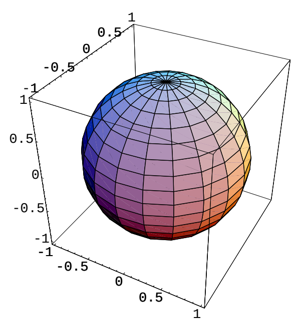
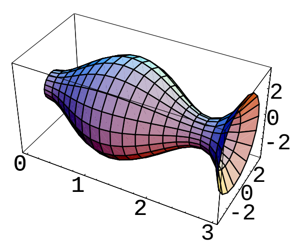
Sample Calculation: The curve $\gamma(s) = \left(\cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$ is parameterized by arc length. Compute $T(s), N(s), B(s)$, and $\tau(s)$.

2 Surfaces

In this course, surfaces are given by a map $s(u,v)$ from the coordinate plane to R^3. If $z = f(u,v)$, we define $s(u,v) = (u, v, f(u,v))$. Below are examples when f is u^2+v^2, $-u^2+v^2$, v^2, and $\sqrt{1-u^2-v^2}$.

We are also interested in the sphere written parametrically as $s(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$, and in a surface of revolution written parametrically as $s(x,\theta) = (x, f(x)\cos\theta, f(x)\sin\theta)$.

The most crucial idea of the entire course is that we can do calculations in the plane (u,v), but arrange the calculations so they tell us something about the three dimensional surface.

3 Tangent Vectors, Dot Products, g_{ij}

Let $g_{11} = \frac{\partial s}{\partial u}\cdot\frac{\partial s}{\partial u}$, $g_{12} = \frac{\partial s}{\partial u}\cdot\frac{\partial s}{\partial v}$, $g_{22} = \frac{\partial s}{\partial v}\cdot\frac{\partial s}{\partial v}$. In the special case that $s(u,v) = (u,v,f(u,v))$ these formulas become $g_{11} = 1 + \left(\frac{\partial f}{\partial u}\right)^2$, $g_{12} = \frac{\partial f}{\partial u}\frac{\partial f}{\partial v}$, $g_{22} = 1 + \left(\frac{\partial f}{\partial v}\right)^2$. Using the g_{ij}, we can compute dot products of vectors, lengths of vectors, and lengths of curves by computing in two dimensional local coordinates, rather than jumping up to the third dimension. The g_{ij} can be found by a two-dimensional person on the surface using a very small ruler and a very small protractor.

Here are the details. Every vector $X = (a, b)$ in the coordinate plane corresponds to a vector $\tilde{X} = a\frac{\partial s}{\partial u} + b\frac{\partial s}{\partial v}$ in R^3. We can compute fancy dot products $\Big\langle X, Y \Big\rangle$ in the plane or ordinary dot products $\tilde{X} \cdot \tilde{Y}$ in R^3; both calculations give the same result. We can compute fancy lengths of projected curves in the plane or ordinary lengths of curves in R^3; both calculations give the same result. In particular

$$X = (a, b) \qquad \leftrightarrow \qquad \tilde{X} = \left(a\frac{\partial s}{\partial u} + b\frac{\partial s}{\partial v} \right)$$

$$\Big\langle X, Y \Big\rangle = g_{11}ac + g_{12}(ad + bc) + g_{22}bd \qquad \leftrightarrow \qquad \tilde{X} \cdot \tilde{Y} = \tilde{X}_1\tilde{Y}_1 + \tilde{X}_2\tilde{Y}_2 + \tilde{X}_3\tilde{Y}_3$$

$$\int_a^b \sqrt{g_{11}\left(\frac{du}{dt}\right)^2 + 2g_{12}\frac{du}{dt}\frac{dv}{dt} + g_{22}\left(\frac{dv}{dt}\right)^2}\, dt \qquad \leftrightarrow \qquad \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Exercise: Suppose $f(x, y) = x^2 + xy$ and $p = (1, 1, 2)$. Convert this into a parameterized surface $s(u, v)$. Let $X = (1, 1)$ and $Y = (2, 3)$. Compute the corresponding $\tilde{X}$ and $\tilde{Y}$. Compute g_{ij}. Compute $\Big\langle X, Y \Big\rangle$ and $\tilde{X} \cdot \tilde{Y}$ and show that they are equal. Find the length of the curve $(t, t, 2t^2)$ for $0 \le t \le 1$ using the two-dimensional formula and using the three-dimensional formula, and show that the integrals are the same (don't integrate).

Remark: A *geodesic* is a curve of constant speed which locally minimizes the distance between its points. Geodesics satisfy the differential equation

$$\frac{d^2\gamma_k}{dt^2} + \sum_{ij} \Gamma_{ij}^k \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0$$

where

$$\Gamma_{ij}^k = \frac{1}{2}\sum_l (g^{-1})_{il} \left\{ \frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right\}$$

Remark: You may have to calculate some Christoffel symbols and produce these differential equations, as on previous exams, but on the final I will not ask for a derivation of the geodesic equation.

Exercise: Consider a surface of revolution parameterized by $(x, f(x)\cos t, f(x)\sin t)$. Find formulas for the g_{ij}. Then compute the Γ_{ij}^k and write down the differential equations. Show that curves with constant θ are geodesics. Show that curves with constant x are geodesics if and only if $f'(x) = 0$. Interprete this geometrically. (On an examination, I'd probably give you the Γ_{ij}^k.)

Exercise: Use symmetry arguments to determine the geodesics on the sphere.

Exercise: Use symmetry arguments to determine some, but not all, geodesics on the saddle $z = -x^2 + y^2$.

4 Differentiating Functions; Differentiating Tangent Vectors

Let g be a function defined on a surface. For instance, g might give the temperature at each point of the surface. If X is a vector tangent to the surface, then $X(g)$ is the derivative of g in the direction X. This $X(g)$ measures how fast g changes as we move in the direction X.

Similarly, let U assign a three-dimensional vector (U_1, U_2, U_3) to each point of the surface. This vector need not be tangent to the surface. Then $X(U)$ is the derivative of U in the tangent direction X; it measures how the vector U changes as we move along the surface in the direction X.

We usually compute these derivatives in coordinates as follows. If $g(x, y, z)$ is a function on R^3, then the resulting function in coordinates can be obtained by using $s(u, v)$ to write x, y and z as functions of u and v, and forming $g\Big(x(u, v), y(u, v), z(u, v)\Big)$. Suppose the tangent vector X equals (a, b). Then in two dimensions we have

$$X(g) = \left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}\right) g\Big(x(u, v), y(u, v), z(u, v)\Big)$$

To compute the derivative of a three dimensional vector $U = (U_1, U_2, U_3)$, we convert each of the three coordinate functions separately to a function of u and v and differentiate each function. Notice that the result is another three dimensional vector.

Exercise: Suppose our surface is $z = f(x, y) = xy^3$ and we are interested in the point $p = (1, 1, 1)$. If $X = (3, 4)$ and $g = x + y + z^2$, compute $X(g)$.

Exercise: Let U be the vector field $U = (x, y, 0)$ on the unit sphere. Parameterize the sphere using spherical coordinates as usual. Let $X = \frac{\partial}{\partial \phi} = (0, 1)$. Compute $X(U)$.

Exercise: Let $U = n$ be the unit normal field to the sphere. Compute U. Then compute $X(U)$ where $X = \frac{\partial}{\partial \phi} = (0, 1)$. Show that $X(U)$ is a multiple of X.

Remark: Consider the special case when $U = Y$ is also tangent to the surface. To compute $X(Y)$, we write X as a two dimensional vector, *but we must write Y as the three dimensional vector $\tilde{Y}$*. The resulting vector $X(Y)$ will thus be a three dimensional vector field, and need not be tangent to the surface. For instance, suppose our surface is a sphere and suppose the vectors U circle the sphere horizontally. The derivative of U points inward toward the z-axis rather than tangent to the sphere.

Example: Parameterize $z = x^2 + y^2$ in polar coordinates as $s(r, \theta) = (r\cos\theta, r\sin\theta, r^2)$. Let $X = (1, 0) = \frac{\partial}{\partial r}$ and let $Y = (0, 1) = \frac{\partial}{\partial \theta}$. Let us compute $X(Y)$. We must write Y as a *three dimensional* vector. Notice that Y corresponds to $\frac{\partial s}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$. We then differentiate this with respect to $X = \frac{\partial}{\partial r}$, obtaining $X(Y) = (-\sin\theta, \cos\theta, 0)$. The answer is a three dimensional vector.

5 The Fundamental Decomposition

If $s(u, v)$ is a surface, the vectors $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$ are tangent, so their cross product is perpendicular to the surface. When this cross-product is divided by its length, we get the unit normal field n:

$$n = \frac{\frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v}}{\left|\left|\frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v}\right|\right|}$$

Using this normal vector, we can decompose any vector U into a normal component and a tangential component by writing

$$U = \Big(U \cdot n\Big)n + \Big(U - (U \cdot n)n\Big)$$

In particular, if X and Y are tangent vector fields, we can decompose $X(Y)$ and $X(n)$ into normal and tangential components. This gives the following fundamental decomposition:

$$\begin{aligned} X(Y) &= b(X, Y)n + \quad \nabla_X Y \\ X(n) &= \qquad\qquad\qquad B(X) \end{aligned}$$

These equations define a real-valued function $b(X, Y)$ defined on pairs of tangent vectors, a tangent-vector valued function $\nabla_X Y$, and a linear map B from tangent vectors to tangent vectors. The object b is often called the *second fundamental form* and the object $\nabla_X Y$ is called the *covariant derivative.*

Exercise Consider the surface $z = x^2 + y^2$. Parameterize this surface using polar coordinates as $s(r, \theta) = (r\cos\theta, r\sin\theta, r^2)$. Compute n. Decompose $U = (x, y, 0)$ into normal and tangential components. Draw a picture to show that your decomposition is reasonable.

Let $X = (1, 0) = \frac{\partial}{\partial r}$ and $Y = (0, 1) = \frac{\partial}{\partial \theta}$. Compute $X(Y)$ and $Y(X)$. Decompose $X(Y)$ and $Y(X)$ into normal and tangential components. Decompose $X(n)$ into normal and tangential components.

Using the results of the previous calculation, find $\nabla_X(Y)$. This is a tangent vector so it must be a linear combination of X and Y. Find this combination. Compute $B(X)$ and $b(X,Y)$.

Remark: For more examples, consult the cheat sheet calculating such derivatives and decompositions on the sphere. This cheat sheet will be printed at the end of the final examination so you can consult it during the exam.

Remark: We will need two facts about this decomposition much later on. First, b and B are closely related; indeed

$$b(X,Y) = - < B(X), Y >$$

This follows because n and Y are perpendicular so $< n, Y >= 0$ and

$$0 = X < n, Y >=< X(n), Y > + < n, X(Y) >=< B(X), Y > + < n, b(X,Y)n + \nabla_X Y >$$

$$=< B(X), Y > + b(X,Y).$$

Second, B is a symmetric matrix: $< B(X), Y >=< X, B(Y) >$. This holds because $[X,Y] = X(Y) - Y(X) = b(X,Y)n + \nabla_X Y - b(Y,X)n - \nabla_Y X$; since $[X,Y]$ is another tangent vector, the normal components of the two sides are

$$0 = b(X,Y)n - b(Y,X)n$$

Then $< B(X), Y >= -b(X,Y) = -b(Y,X) =< B(Y), X >=< X, B(Y) >$. The truth is that we only use this result indirectly by way of a theorem in linear algebra. This theorem asserts that we can find an orthonormal basis of T_p consisting of eigenvectors of B. I'll never ask you for a proof, but you might notice the moment that we use this in the proof of the Theorema Egregium.

6 Curvature; Extrinsic Geometry

The next part of our couse is about measuring the principal curvatures of a surface. The basic idea is quite straightforward; in outline

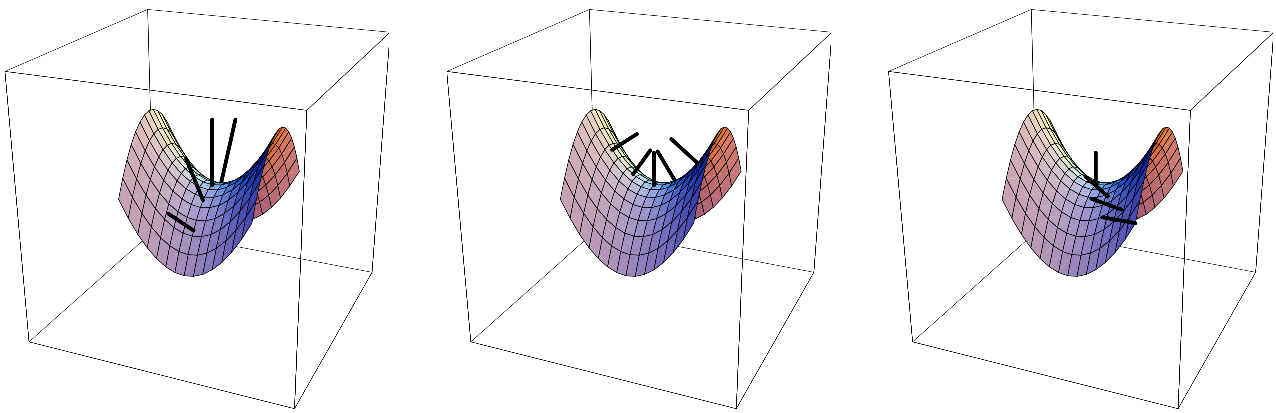
- geometric intuition suggests that our surface is curved just in case n changes as we move from point to point; this change is measured by $X(n) = B(X)$; further examples suggest that the principal curvatures should be the eigenvalues of the matrix B

- thus we can compute the principal curvatures systematically; start by letting e_1 and e_2 be the basis vectors $e_1 = \frac{\partial}{\partial u}$ and $e_2 = \frac{\partial}{\partial v}$ for the tangent space

- compute $e_1(n)$ and $e_2(n)$; each answer will be another tangent vector and thus a linear combination of e_1 and e_2

- let B be the matrix formed by the coefficients of these combinations
- the principal curvatures κ_1 and κ_2 are the eigenvalues of this matrix

Here are a few more details. Consider $z = -x^2 + y^2$. In the first picture below, we move in the x-direction and the change of n is also in this direction. A calculation shows that in fact $B(X) = 2X$. In the second picture we move in the y-direction and the change of n is in the negative y-direction. A calculation shows that $B(Y) = -2Y$.



Finally, in the third picture we move in an unusual direction T and $B(T)$ is not a multiple of T. Let us measure curvature so the sign of upward curvature is positive. Then the curvature in the X-direction should be negative, perhaps -2, and the curvature in the Y-direction should be positive, perhaps 2.

Definition: The eigenvectors of B are called the *principal directions*; the negatives of the corresponding eigenvalues are called the *principal curvatures*.

Remark: We introduce notation to be used from now on. The vector $(1, 0)$ will be written $\frac{\partial}{\partial u}$ and the vector $(0, 1)$ will be written $\frac{\partial}{\partial v}$. Thus

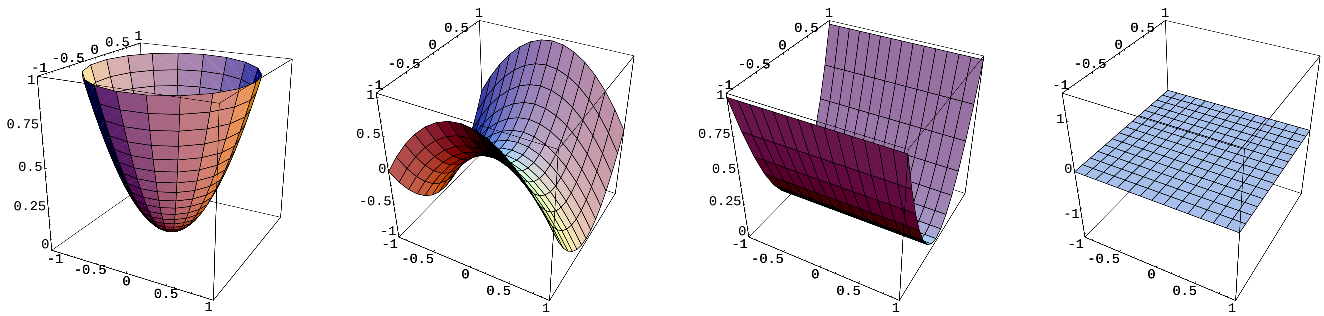
$$(a, b) = a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}$$

Exercise: Suppose $s(u, v) = (u, v, f(u, v))$ and both partial derivatives of f vanish at the origin. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Show that the matrix for B at the origin is

$$-\frac{1}{2}\begin{pmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial x \partial y} \\ \frac{\partial^2 g}{\partial x \partial y} & \frac{\partial^2 g}{\partial y^2} \end{pmatrix}$$

Exercise: In particular, if $f = f(0,0) + \frac{\kappa_1}{2}x^2 + \frac{\kappa_2}{2}y^2 + \dots$, then the principal curvatures at the origin are κ_1 and κ_2.

Summary: The outcome of the material in this section is that any surface can be approximated near p by one of the following quadratic surfaces, and we can determine which surface once we know κ_1 and κ_2



Exercise: Compute κ_1 and κ_2 at an arbitrary point of the surface $z = x^2 + y^2$, which is best studied in polar coordinates $s(r, \theta) = (r\cos\theta, r\sin\theta, r^2)$. By symmetry, κ_1 and κ_2 should only depend on r. As $r \to \infty$, both values should go to zero; explain why.

7 The Covariant Derivative

Next we examine the covariant derivative $\nabla_X Y$. This derivative turns out to be intrinsic; it can be computed by a two-dimensional person on the surface without leaving the surface. Notice that calculating $X(Y)$ requires us to leave the surface, so the intrinsic nature of $\nabla_X Y$ is not obvious from the formula $X(Y) = b(X,Y)n + \nabla_X Y$.

To prove that $\nabla_X Y$ is intrinsic, we find a formula for this derivative depending only on the Christoffel symbols, and thus ultimately on g_{ij}. Indeed suppose that $X = (1,0) = \frac{\partial}{\partial u}$. In coordinates $Y = (Y_1, Y_2)$ and we could easily guess that

$$\nabla_{\frac{\partial}{\partial u}}(Y_1, Y_2) = \left(\frac{\partial Y_1}{\partial u}, \frac{\partial Y_2}{\partial u}\right)$$

But this turns out to be wrong! The correct formulas are

$$\nabla_{\frac{\partial}{\partial u}}(Y_1, Y_2) = \left(\frac{\partial Y_1}{\partial u} + \Gamma^1_{11}Y_1 + \Gamma^1_{12}Y_2, \frac{\partial Y_2}{\partial u} + \Gamma^2_{11}Y_1 + \Gamma^2_{12}Y_2\right)$$

$$\nabla_{\frac{\partial}{\partial v}}(Y_1, Y_2) = \left(\frac{\partial Y_1}{\partial v} + \Gamma^1_{21}Y_1 + \Gamma^1_{22}Y_2, \frac{\partial Y_2}{\partial v} + \Gamma^2_{21}Y_1 + \Gamma^2_{22}Y_2\right)$$

and for general X, $\nabla_X Y$ is a linear combination of these. For instance, if $X = (3, 4)$, then $\nabla_X Y$ is 3 times the first formula plus 4 times the second.

Exercise: Consider the surface $z = x^2 + y^2$ written as always in polar coordinates. Let $X = \frac{\partial}{\partial r}$ and $Y = \frac{\partial}{\partial \theta}$. An exercise on page five asked you to compute $X(Y)$ and its decomposition into $b(X, Y)n + \nabla_X Y$. Compute $\nabla_X Y$ intrinsically using the above formula, and show that you get the same result that you got on page five.

Remark: To obtain the above formulas, we proceeded axiomatically. We argued that $\nabla_X Y$ should behave more or less like derivatives of vector fields in several variable calculus. This gave a list of axioms

- $\nabla_X Y$ is linear in X
- $\nabla_X Y$ is linear in Y
- $\nabla_X fY = X(f)Y + f\nabla_X Y$
- $X\Big\langle Y, Z\Big\rangle = \Big\langle \nabla_X Y, Z\Big\rangle + \Big\langle Y, \nabla_X Z\Big\rangle$
- $\nabla_X Y - \nabla_Y X = [X, Y]$

We then showed that these axioms imply the formulas given at the top of page 8. There are many details in this argument which you can skip, but I want you to know the central argument, which proceeds as follows:

Apply the third axiom to obtain

$$X\Big\langle Y, Z\Big\rangle + Y\Big\langle Z, X\Big\rangle - Z\Big\langle X, Y\Big\rangle$$

$$= \Big\langle \nabla_X Y, Z\Big\rangle + \Big\langle Y, \nabla_X Z\Big\rangle + \Big\langle \nabla_Y Z, X\Big\rangle + \Big\langle Z, \nabla_Y X\Big\rangle - \Big\langle \nabla_Z X, Y\Big\rangle - \Big\langle X, \nabla_Z Y\Big\rangle$$
$$= 2\Big\langle \nabla_X Y, Z\Big\rangle + \Big\langle [Y, X], Z\Big\rangle + \Big\langle [X, Z], Y\Big\rangle + \Big\langle [Y, X], Z\Big\rangle$$

In this formula, substitute $X = \frac{\partial}{\partial u_i}, Y = \frac{\partial}{\partial u_j}, Z = \frac{\partial}{\partial u_k}$. Since mixed partial derivatives are equal, bracket products are zero and the formula reads

$$X\Big\langle Y, Z\Big\rangle + Y\Big\langle Z, X\Big\rangle - Z\Big\langle X, Y\Big\rangle = \frac{\partial g_{jk}}{\partial u_i} + \frac{\partial g_{ik}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_k}$$

$$= 2\Big\langle \nabla_X Y, Z\Big\rangle = 2\Big\langle \nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_k}\Big\rangle$$

The expression $\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j}$ must be a linear combination of basis vectors. Write it as

$$\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = \Gamma_{ij}^1 \frac{\partial}{\partial u_1} + \Gamma_{ij}^2 \frac{\partial}{\partial u_2}$$

where for a moment we are *not assuming* that these are our old Christoffel symbols. Then

$$2\Big\langle \nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_k} \Big\rangle = 2 \sum \Gamma_{ij}^l g_{lk}$$

and so

$$\sum \Gamma_{ij}^l g_{lk} = \frac{1}{2} \left\{ \frac{\partial g_{jk}}{\partial u_i} + \frac{\partial g_{ik}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_k} \right\}$$

We conclude that the Γ_{ij}^k satisfy the same formulas as the Christoffel symbols which appeared in the geodesic equation. So these Christoffel symbols are also the key ingredient in our intrinsic formula for $\nabla_X Y$.

8 The Curvature Tensor and the Theorema Egregium

We now come to the grand climax of the course. We are going to define a quantity κ called the *Gaussian curvature* and show that a two-dimensional person can compute this number. We will prove that $\kappa = \kappa_1 \kappa_2$ if our surface is embedded in R^3. Consequently a two-dimensional person can determine whether the surface looks locally like a paraboloid, like a saddle, or like a flat plane.

We begin with Riemann's counting argument which suggests that there is just one magical function which determines the geometry of a surface near a point. To give a geometry, we must choose coordinates and then determine the three functions g_{11}, g_{12}, and g_{13}. But these numbers depend on the coordinate choice; we would have obtained different functions if we had chosen different coordinates $\tilde{u} = \psi(u, v)$ and $\tilde{(v)} = \phi(u, v)$. So there are three g_{ij} and two degrees of freedom. Since

$$3 - 2 = 1,$$

there should be one piece of truly geometric information hidden in the g_{ij}.

The tool we develop to reveal this hidden information ultimately comes from the theorem in several variable calculus asserting that mixed partial derivatives are equal: $\frac{\partial^2 g}{\partial u_i \partial u_j} = \frac{\partial^2 g}{\partial u_j \partial u_i}$. If X and Y are vector fields and we compute directional derivatives of a function g using them, we find that the operators X and Y do not commute. But equality of mixed partial derivatives shows that the operators *almost commute*; in the expression

$$[X, Y]g = X(Y(g)) - Y(X(g))$$

all the second order derivatives cancel out and the difference is given by taking a directional derivative in a new tangent direction $[X, Y]$.

Exercise: Let $X = (u, 1)$ and $Y = (0, u)$. Compute $X(Y(g))$ and $Y(X(g))$ and use the computation to show that $[X, Y] = (0, u) = Y$.

Consequently, when dealing with derivatives of objects with respect to vector fields X, we expect some mild noncommutativity and deal with it by adding terms of the form $[X, Y]$ to our formulas. Here are four such formulas

- $X(Y(g)) - Y(X(g)) = [X, Y](g)$
- $X(Y(U)) - Y(X(U)) = [X, Y](U)$
- $\nabla_X Y - \nabla_Y X = [X, Y]$
- $\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z$

And now the main point. *The first three formulas are true. But the final formula is not quite true, and the exception is exactly the secret geometric information* κ

Definition: Let X, Y, Z, and W be tangent vector fields. The Riemann Curvature Tensor is the gadget

$$R(X, Y, Z, W) = \left\langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W \right\rangle$$

which measures the failure of the last formula in the previous list.

Remark: This is a famous gadget; whenever someone talks about the Gaussian curvature, or the curvature of the universe, or the distortion of light in space time, they are talking about this object. In class we listed a number of properties of R. None of these properties are difficult to prove. I don't want you to know the proofs, but I do want you to know the list of properties:

Theorem 1 *The curvature tensor R has the following properties:*

- *R is linear over functions in each variable separately if the other variables are held fixed, so for instance*

$$R(X_1 e_1 + X_2 e_2, Y, Z, W) = X_1 R(e_1, Y, Z, W) + X_2 R(e_2, Y, Z, W)$$

- *Consequently*

$$R\left(\sum_i X_i e_i, \sum_j Y_j e_j, \sum_k Z_k e_k, \sum_l W_l e_l\right) = \sum_{ijkl} R_{ijkl} X_i Y_j Z_k W_l$$

  where $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ *So the curvature tensor depends on* $2 \times 2 \times 2 \times 2 = 16$ *pieces of information.*

- *R is antisymmetric in the first two and last two variables, so*

$$R(Y, X, Z, W) = -R(X, Y, Z, W)$$

$$R(X, Y, W, Z) = -R(X, Y, Z, W)$$

- *Consequently the only* R_{ijkl} *which are nonzero are*

$$R_{1212} = -R_{2112} = R_{2121} = -R_{1221}$$

  and the curvature tensor depends on only one piece of information.

- *Let* $\{f_1, f_2\}$ *be an orthonormal basis of the tangent space at a point* p*. Then the number*

$$R(f_1, f_2, f_1, f_2)$$

  does not depend on the choice of this basis, and thus is an invariant number depending only on the geometry near p*. We call the negative of this number the* Gaussian Curvature κ*, so by definition*

$$R(f_1, f_2, f_1, f_2) = -\kappa$$

- *If* $e_1 = \frac{\partial}{\partial u}$ *and* $e_2 = \frac{\partial}{\partial v}$ *is the standard basis for the tangent space at* p *and* g_{ij} *are the related metric tensor coefficients, then*

$$R_{1212} = R(e_1, e_2, e_1, e_2) = \det(g_{ij}) R(f_1, f_2, f_1, f_2).$$

  Consequently the number κ *completely determines* R*.*

Remark: Thus the curvature tensor produces exactly what we wanted: a single invariant number at p which depends only on the geometry near p.

Finally we come to Gauss' great theorem, and this time I definitely want you to know the proof:

Theorem 2 (Theorema Egregium) *If* $\mathcal{S}$ *is a surface embedded in* R^3*, then*

$$-R(f_1, f_2, f_1, f_2) = \kappa = \kappa_1 \kappa_2$$

Proof: Let X, Y, Z, and W be four tangent vector fields. If we compute derivatives in R^3 by just differentiating each component separately, then X and Y commute up to $[X, Y]$ since $[X, Y]$ tells the full story for derivatives of functions. So

$$X(Y(Z)) - Y(X(Z)) = [X, Y](Z)$$

Now decompose the resulting derivatives into normal and tangential components:

$$X\Big(b(Y, Z)n + \nabla_Y Z\Big) - Y\Big(b(X, Z)n + \nabla_X Z\Big) - \Big(b([X, Y], Z)n + \nabla_{[X,Y]} Z\Big) = 0$$

Differentiate a final time using the product rule, and decompose once more into normal and tangential components:

$$X\Big(b(Y, Z)\Big)n + b(Y, Z)X(n) + b(X, \nabla_Y Z)n + \nabla_X \nabla_Y Z$$

$$-Y\Big(b(X, Z)\Big)n + b(X, Z)Y(n) + b(Y, \nabla_X Z)n + \nabla_Y \nabla_X Z$$

$$-\Big(b([X, Y], Z)n + \nabla_{[X,Y]} Z\Big) = 0$$

In this expression, throw away the normal components and write down the tangential components:

$$b(Y, Z)X(n) + \nabla_X \nabla_Y Z - b(X, Z)Y(n) - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = 0$$

Take the terms involving b and B to the other side:

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = b(X, Z)Y(n) - b(Y, Z)X(n)$$

Take the inner product of both sides with W and recall that $X(n) = B(X)$:

$$R(X, Y, Z, W) = b(X, Z) < B(Y), W > -b(Y, Z) < B(X), W >$$

Recall that $b(X, Y) = - < B(X), Y >$:

$$R(X, Y, Z, W) = - < B(X), Z >< B(Y), W > + < B(Y), Z >< B(X), W >$$

But B is a symmetric matrix, so we can find an orthonormal basis $\{f_1, f_2\}$ of eigenvectors. The negatives of the eigenvalues are the principal curvatures κ_1 and κ_2. Substitute $X = f_1, Y = f_2$, $Z = f_1, W = f_2$ to obtain

$$R(f_1, f_2, f_1, f_2) = - < B(f_1), f_1 >< B(f_2), f_2 > + < B(f_2), f_1 >< B(f_1), f_2 >$$

and so

$$-\kappa = - < -\kappa_1 f_1, f_1 >< -\kappa_2 f_2, f_2 > + < -\kappa_2 f_2, f_1 >< -\kappa_1 f_1, f_2 >= -\kappa_1 \kappa_2$$

9 Harvesting the Consequences

The rest of the course was about the consequences of all of this for a two dimensional person on a surface. That person knows that there is an invariant κ which determines the geometry of the surface. But so far the invariant has been defined in a complicated algebraic manner via the curvature tensor. Our two dimensional person would like to understand κ geometrically.

Because the course is coming to an end, I cannot expect you to know the proofs of the results which follow. But I definitely want you to know the statements of the main theorems.

The invariant κ is a function, but we gain a lot of insight by studying the special case when it is a constant. Here are the main theorems:

Theorem 3 *If we magnify the surface by M, replacing $< X, Y >$ by $M^2 < X, Y >$, then κ changes to $\frac{\kappa}{M^2}$.*

If follows that after magnification we can assume that a constant κ is 1, 0, or -1.

Theorem 4 *On a sphere of radius one, $\kappa = 1$*

Theorem 5 *On the plane, $\kappa = 0$*

Theorem 6 *On the Poincare Non-Euclidean disk, $\kappa = -1$*

Theorem 7 *If κ equals one, zero, or minus one on a surface $\mathcal{S}$, then locally near any $p \in \mathcal{S}$ the surface is isometric to a small neighborhood of the sphere, the plane, or the non-Euclidean disk, respectively.*

Remark: In summary, there are only three geometries with constant Gaussian curvature, completely determined by κ.

10 More Consequences; Gauss-Bonnet

Theorem 8 (Gauss-Bonnet) *Let $\mathcal{R}$ be a region in a coordinate patch of a surface, with boundary edges traversed counterclockwise. Let $\triangle\theta$ be the exterior angles at corners of this region, κ_g the geodesic curvature along the edges, and κ the Gaussian curvature of the surface. Then*

$$\sum_{\textit{vertices}} \triangle\theta + \sum_{\textit{edges}} \int \kappa_g + \int\int_{\mathcal{R}} \kappa = 2\pi.$$

Theorem 9 *Let $\mathcal{R}$ be a triangle on a surface with interior angles α, β, γ and geodesic sides. Then*

$$\alpha + \beta + \gamma - \pi = \int\int_{\mathcal{R}} \kappa$$

Remark: The theorem just stated is particularly important because it gives a geometric interpretation of κ, even for nonconstant κ.

Theorem 10 (Global Gauss-Bonnet) *Let $\mathcal{S}$ be a compact surface with g holes. Then*

$$\frac{1}{2\pi} \int\int_{\mathcal{S}} \kappa = 2 - 2g$$

Theorem 11 *A compact surface with g holes can be given a metric with constant $\kappa > 0$ if and only if it is a sphere.*

Theorem 12 *A compact surface with g holes can be given a metric with constant $\kappa = 0$ and thus with Euclidean flat geometry if and only if it is a torus.*

Theorem 13 *A compact surface with g holes can be given a metric with constant $\kappa < 0$ if and only if $g \geq 2$.*

Remark: I am hoping that you want to learn a portion of the end of the course in detail rather than learning a lot of little facts for the final. So at the end of the exam, I'll give you a choice of writing about one of the last sections of the course. If you want to concentrate on calculations, you can skim this last question. You may prefer to rush through some calculations, scattering factors of two to the winds, and then give more details in the last question. Here are a few samples, although I'll give more choices:

- How do we calculate κ on the Poincare disk?

- Sketch a portion of the proof that a surface with constant κ is locally isometric to a sphere, plane, or non-Euclidean disk. Explain some step in the proof with enough detail that the technical results are clear.

- Sketch the proof of the Euler-Poincare formula, and explain how the Global Gauss-Bonnet theorem follows from this argument.

- Explain the opening steps of the proof of Gauss-Bonnet, up to a formula which contains $\triangle\theta$ and $\int \kappa_g$.

- Explain the ending steps of the proof of Gauss-Bonnet, in which Stokes formula is applied to convert a line integral to $\int\int \kappa$.

- If $\{f_1, f_2\}$ is an orthonormal basis of T_p, prove that $R(f1, f2, f1, f2)$ is independent of the choice of f_1 and f_2.